Photon Chopping: New Way to Measure the Quantum State of Light

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We propose the use of a balanced $2N$-port as a technique to measure the pure quantum state of a single-mode light field. In our scheme the coincidence signals of simple, realistic photodetectors are recorded at the output of the $2N$-port. We show that applying different arrangements both the modulus and the phase of the coefficients in a finite superposition state can be measured. In particular, the photon statistics can be so measured with currently available devices.

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At least since the Greeks [1] we have tractated the problem: "What happens when matter is divided into smaller and smaller pieces?" The outwardly simple question led to thousands of years of discussion in philosophy and turned out to be extremely fruitful in modern physics [2]. Can we apply the idea of chopping to light? The answer is straightforward. A balanced $2N$-port [3] is a suitable device to split up an incoming light pulse. As a result small splinter pulses appear at the output ports containing only a small fraction of the original pulse energy, in the limit of large N only one photon. These pulses, however, carry valuable information, and as we will show in this Letter by measuring them in coincidence one can reconstruct the complete quantum state of the incoming light mode.

The quantum state measurement problem has attracted increasing interest in the recent years in quantum optics [4]. In particular, the measurement of the Wigner function [5] following the idea of Vogel and Risken [6] became the focus of experimental efforts [7]. Only recently, also schemes for directly reconstructing the density operator from the same experiments were developed [8–10], thus avoiding the detour via the Wigner function. All these proposals are based on the homodyne technique, i.e., the mixing with a strong local oscillator in order to measure quadrature distributions. The set of quadrature histograms is then used to reconstruct different mathematical objects equivalent to the wave function. Although the pioneering experiments of Smithey, Beck, and Raymer [7] are very promising, it is still beyond the scope of experiments to resolve the fine details of a quantum state being characteristic for the nonclassical behavior. For example, the oscillations in the photon number statistics for squeezed states leading to the Schleich-Wheeler oscillations in the limit of large squeezing have not been observed yet.

Our proposal offers a novel approach to the problem avoiding the use of strong, classical fields. In contrast to the homodyne technique we may then apply detectors being sensitive at low intensities. Also, the way of the reconstruction is just complementary to the tomographical method. Instead of discretizing an intrinsically continuous transformation we find a system of equations for a discrete set of variables. In this sense our method is similar to quantum state endoscopy [11], a very recent proposal of how to reconstruct the state of a light field in a cavity.

Let us first focus on the (indirect) measurement of the photon number distribution. Its direct measurement would require high efficiency photodetectors discriminating between $0, 1, 2, \ldots$ photons. Such detectors are, however, not available today. One type of the existing high-efficiency detectors, such as avalanche photodiodes, indicates only the presence of photons due to saturation effects. We will refer to them as type I detectors. More sophisticated detectors which we will call type II detectors discriminate between zero, one, and more than one photon. Unfortunately, this higher discrimination must be paid for by lower detection efficiency.

Let us consider a single-mode field in a *pure state* $|\varphi\rangle$,

$$|\varphi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=0}^{\infty} |c_n| \exp(i\phi_n)|n\rangle, \qquad (1)$$

entering a balanced $2N$-port as shown in Fig. 1. We would like to emphasize that the determination of the photon number distribution, which is our first goal, is not restricted to pure states. One has only to replace $|c_n|^2$ by the diagonal density matrix elements ρ_{nn} in the following considerations. Although any balanced $2N$-port [3,12–14] would serve equally well, we focus, for simplicity, on a $2N$-port corresponding to the unitary transformation

$$\mathbf{U}_N = \mathbf{U}_2 \otimes \mathbf{U}_{N/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{U}_{N/2} & \mathbf{U}_{N/2} \\ \mathbf{U}_{N/2} & -\mathbf{U}_{N/2} \end{pmatrix}, \qquad (2)$$

where

$$\mathbf{U}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \qquad (3)$$

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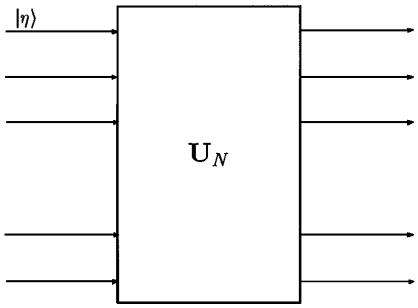

FIG. 1. $2N$-port setup for the determination of the photon number distribution. The unknown state is fed into the first input, and multiple coincidences are measured at the output.

This device has $N = 2^k$ input and output channels. We first assume that type II detectors are placed in the output channels. When we feed the system via the first input (Fig. 1), any input Fock state $|n\rangle$ transforms as

$$\mathbf{U}_N|n\rangle = \mathbf{U}_N \frac{1}{\sqrt{n!}} (\hat{a}_1^\dagger)^n |0\rangle = \frac{1}{\sqrt{n!}} \frac{1}{(\sqrt{N})^n} \left(\sum_{i=1}^{N} \hat{b}_i^\dagger \right)^n |0\rangle , \tag{4}$$

where $\hat{a}_i^\dagger$ and $\hat{b}_i^\dagger$ are the creation operators for the ith input and output modes, respectively. Thus the output state is the following linear combination of multimode number states:

$$\mathbf{U}_N|n\rangle = \sqrt{\frac{n!}{N^n}} \sum_{k_1+k_2+\cdots+k_N=n} \frac{1}{\sqrt{k_1!\, k_2! \cdots k_N!}} \times |k_1, k_2, \ldots, k_N\rangle . \tag{5}$$

This implies that the joint probability $P_n(k_1, k_2, \ldots, k_N)$ of finding k_i photons at the ith output ($i = 1, 2, \ldots, N$) follows the statistics of distinguishable particles

$$P_n(k_1, k_2, \ldots, k_N) = \frac{n!}{N^n} \frac{1}{k_1!\, k_2! \cdots k_N!} . \tag{6}$$

The detectors are supposed to indicate the presence of zero, one, or more photons; therefore we can measure the statistics of the coincident events of n detectors giving simultaneously a one-photon signal. To this case only one possible input Fock state corresponds, namely, that with photon number n. The total probability of such coincident events is a sum of $P_n(k_1, k_2, \ldots, k_N)$ on condition that n of the indices $\{k_i\}$ equal 1, and all the others are zero. This yields the probability w_n^N of n coincident one-photon clicks for an arbitrary input state $|\varphi\rangle$

$$w_n^N = \binom{N}{n} \frac{n!}{N^n} |c_n|^2 . \tag{7}$$

Utilizing this simple result, we can readily infer the photon number distribution $|c_n|^2$ in the signal field, however, only up to $n = N$. Hence it is desirable to have a device at one's disposal that has as many output ports as possible. Note that for $n \ll N$, w_n^N approaches $|c_n|^2$.

Let us now turn to the physically more interesting case of type I detectors. Then the relationship between the measured coincidence probabilities and the photon distribution $|c_n|^2$ becomes more involved. In fact, n incoming photons can now trigger m simultaneous clicks, where $m = 1, 2, \ldots, n$. The corresponding probability is given by

$$P_{m,n}^N = \frac{n!}{N^n} \sum_{k_1+k_2+\cdots+k_N=n}^{(m)} \frac{1}{k_1!\, k_2! \cdots k_N!} , \tag{8}$$

where (m) refers to the summation condition that exactly m of the indices $\{k_i\}$ are nonzero, and the index N refers to the N output ports. The probabilities (8) satisfy the following recursion relation:

$$P_{m,n+1}^N = \frac{1}{N} [m P_{m,n}^N + (N - m + 1) P_{m-1,n}^N] . \tag{9}$$

Here, the two terms on the right-hand side reflect the fact that an extra photon can be directed either to an already occupied channel or to an until then empty channel. Evaluating the sum in Eq. (8) we find the explicit form of $P_{m,n}^N$:

$$P_{m,n}^N = \frac{1}{N^n} \binom{N}{m} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m - i)^n , \quad N \geq n \geq m ; \tag{10}$$

for $n < m$, $P_{m,n}^N$ is zero. It can be easily verified that this expression satisfies the recursion relation (9). Considering now the observable probability that m detectors click, w_m^N, we see that all Fock states $|n\rangle$ with $n \geq m$ will contribute. However, a one-to-one correspondence between w_m^N and the photon number distribution $|c_n|^2$ exists only when the latter can actually be truncated at $n = N$. Practically, this means that the requirement of a large number of output ports is now even more serious than in the previous case. When the mentioned truncation can be justified, we have

$$w_m^N = \sum_{n=m}^{N} P_{m,n}^N |c_n|^2 . \tag{11}$$

As shown above, the probabilities $P_{m,n}^N$ form an $N \times N$ upper triangular matrix. Its inversion yields a matrix of the same type. The construction of the inverse matrix is simple, it is equivalent to solving the system of N linear equations represented by Eq. (11). This can be done using a recursion starting from the last equation, which contains only one unknown parameter. Thus we find the recursion relation for the inverse matrix (for $k \neq 0$)

$$(P^N)_{n,n+k}^{-1} = - \frac{1}{P_{n+k,n+k}^N} \sum_{j=0}^{k-1} (P^N)_{n,n+j}^{-1} P_{n+j,n+k}^N \tag{12}$$

and $(P_{n,n}^N)^{-1} = 1/P_{n,n}^N$. For instance, the inverse matrix for $N = 4$ reads

$$(P^4)^{-1} = \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & -1 \\ 0 & \frac{4}{3} & -2 & \frac{22}{3} \\ 0 & 0 & \frac{8}{3} & -16 \\ 0 & 0 & 0 & \frac{32}{3} \end{pmatrix} . \tag{13}$$

The photon number distribution is then uniquely related to the measured multiple coincidences in the form

$$|c_n|^2 = \sum_{m=n}^{N} (P^N)^{-1}_{n,m} w^N_m. \tag{14}$$

We note that the generalization to a mixed state is straightforward. The transformation (14) yields then the main diagonals of the density matrix.

In the preceding analysis it was assumed that the detectors have unit efficiency. Unfortunately, this is not the case in practice. However, the effect of nonunit detection efficiency in the two measurement schemes is readily taken into account. In fact, it became obvious from the above analysis that in those schemes the photons behave like classical (distinguishable) particles, and it is well known that detector inefficiency can be modeled by an absorber (or a partly transmitting mirror) placed in front of the detector. Since the removal of photons by such an absorber is a random process, it does not matter whether the damping occurs behind or before the $2N$-port. This means we can equally model the detector inefficiency by placing just one absorber in the signal before it impinges on the $2N$-port. Hence what we actually reconstruct under realistic conditions is the photon distribution of a *damped* signal. From that data the true photon distribution can be found by inverting a Bernoulli transformation. Actually this has been analyzed only recently [15]. Further, we would like to mention that for the above-mentioned type of experiments a beam splitting device with the desired large number of output ports actually exists. The needed $2N$ ports can be constructed using beam splitters and phase shifters. For the given arrangement we need at most N beam splitters [16,17]. An excellent candidate for a $2N$-port is also a plate beam splitter [18].

In the second part of this paper we will show how our $2N$-port enables us also to determine the phases ϕ_n of the coefficients c_n with the help of detectors of the second type. In this part we limit ourselves to an idealized situation with no losses and ideal detectors. Moreover, the assumption of a pure initial state (1) is now essential. However, even with these limitations the problem is from a theoretical point of view very interesting.

To this end we will consider the setup shown in Fig. 2. We use the same balanced $2N$-port as before [corresponding to the transformation (2)], but we feed, in addition, into one of the inputs a coherent state $|\alpha\rangle$

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha_n |n\rangle = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{[|\alpha| \exp(i\varphi)]^n}{\sqrt{n!}} |n\rangle \equiv \sum_{n=0}^{\infty} \alpha_n |n\rangle. \tag{15}$$

For simplicity, we suppose the coherent state was sent into the device via the port $N/2 + 1$. However, any other free input would serve the purpose equally well. Note that the

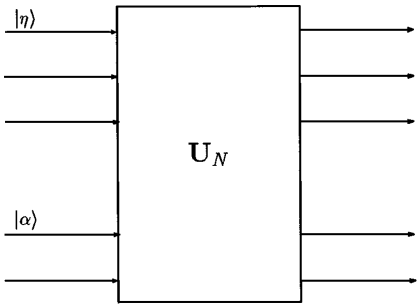

FIG. 2. $2N$-port setup for the determination of the phases of the coefficients of the unknown state. The latter is fed into the first input and the reference coherent state $|\alpha\rangle$ into the input $N/2 + 1$.

input creation operator $\hat{a}^\dagger_{N/2+1}$ transforms as

$$\mathbf{U}_N \hat{a}^\dagger_{N/2+1} |0\rangle = \frac{1}{\sqrt{N}} \left(\sum_{i=1}^{N/2} \hat{b}^\dagger_i - \sum_{j=N/2+1}^{N} \hat{b}^\dagger_j \right) |0\rangle. \tag{16}$$

As before we ask what is the probability to find at the output a multiple coincidence $w^N_n(i, j, \ldots, m)$, where the indices $(i, j, \ldots, m)$ refer to particular detectors. In contrast to the previous case it is of importance which detectors respond. The total input state now reads

$$|\eta\rangle |\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle_1 \otimes \sum_{m=0}^{\infty} \alpha_m |m\rangle_{N/2+1} \otimes |0\rangle_{\text{others}}. \tag{17}$$

The output state $|\psi_{\text{out}}\rangle$ is easily obtained from the relations (4) and (16)

$$|\psi_{\text{out}}\rangle = \sum_{n=0}^{\infty} c_n \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{N}^n} \left(\sum_{i=1}^{N} \hat{b}^\dagger_i \right)^n \otimes \sum_{m=0}^{\infty} \alpha_m \frac{1}{\sqrt{m!}} \frac{1}{\sqrt{N}^m} \left(\sum_{i=1}^{N/2} \hat{b}^\dagger_i - \sum_{N/2+1}^{N} \hat{b}^\dagger_i \right)^m |0\rangle. \tag{18}$$

Using this expression we find the probability for the multiple coincidence $w^N_n(i, j, \ldots, m)$ to be given by

$$w^N_n(i, j, \ldots, m) = \frac{1}{N^n} \times \left| \sum_{k=0}^{n} f_{kn}(i, j, \ldots, m) \sqrt{(n-k)!\, k!} \binom{n}{k} c_{n-k} \alpha_k \right|^2, \tag{19}$$

where the coefficients f_{kn} depend on the combination $i, j, \ldots, m$. In this case we limit ourselves to coincidences among the first $N/2$, all coefficients f_{kn} are equal to 1. Hence the probability (19) no longer depends on the individual detectors that give clicks. So we can easily pass from Eq. (19) to the total probability for n clicks, irrespective of which of the detectors respond

$$\tilde{w}^N_n = \binom{N/2}{n} \frac{1}{N^n} \left| \sum_{k=0}^{n} \sqrt{(n-k)!\, k!} \binom{n}{k} c_{n-k} \alpha_k \right|^2. \tag{20}$$

Now to find the phases ϕ_n of the coefficients c_n is straightforward. Starting with the probability for just one click to occur

$$\tilde{w}_1^N = \tfrac{1}{2}|c_1\alpha_0 + c_0\alpha_1|^2, \tag{21}$$

we have to solve this equation for ϕ_1. Here we assume that the amplitudes $|c_n|$ are already determined from a previous measurement of the photon number distribution $|c_n|^2$. Moreover, we can put $\phi_0 = 0$ without loss of generality, since it is well known that any state is defined only up to an overall phase factor. Then ϕ_1 is the only unknown quantity. However, it should be noted that it cannot be determined from Eq. (21) uniquely, since the latter gives us only $\cos(\phi_1 - \varphi)$. To remove the ambiguity, we will have to perform a second series of measurements with a coherent state of different phase, e.g., $\varphi + \frac{\pi}{2}$. Taking then the expression (20) for $n = 2$ and inserting, in particular, the previously determined value for ϕ_1, one can calculate ϕ_2. In this way one can proceed until all phases ϕ_n are found.

In conclusion, we have shown that a balanced $2N$-port provides an alternative technique, compared to optical homodyne tomography, to determine with the help of realistic photodetectors the quantum state of light and, in particular, the photon number distribution. For the determination of the photon statistics we do not need to use a local oscillator and so we are able to determine the diagonal density matrix elements of a general single-mode light field. A homodyne technique in the absence of a proper (frequency adjusted) laser field (local oscillator) is not able to accomplish such a task. In this respect the proposed method is superior.

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