Transition from Poisson Regularity to Chaos in a Time-Reversal NonInvariant System

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In a random matrix model, the transition from arbitrary to chaotic correlations is analytically evaluated for the case of broken time-reversal invariance. The use of the supersymmetry method allows for a nonperturbative calculation. As an application, the transition from Poisson regularity to chaos is worked out. The result is a double integral which is exact for the entire transition.

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The transition from regular to chaotic behavior is of particular interest for chaos theory. Among the quantum systems showing this crossover, the hydrogen atom in a strong magnetic is probably the most intensely studied one [1]. The transition manifests itself in the spectral fluctuation properties, especially in the nearest neighbor spacing distribution and the spectral rigidity [2,3]. There are many more examples in physics. In heavy ion reactions, a spreading of the electrical quadrupole transition strength has been observed which can be understood in terms of a regularity chaos transition [4]. In condensed matter physics, the phenomenon of localization can also be related to this crossover. Billiard systems [5] show similar transitions as well. Starting from a regularly shaped billiard, slight changes of the geometry can make the dynamics gradually or even abruptly chaotic. Although the list of examples is far from being complete, it is obvious that a deeper theoretical understanding of this transition is a worthwhile task.

To construct a statistical model, we rely on random matrix theory [6]. Because of the general symmetry constraints, a time-reversal invariant system with conserved or broken rotation invariance is modeled by the Gaussian orthogonal (GOE) or symplectic ensemble (GSE), respectively, while the Gaussian unitary ensemble (GUE) models the fluctuation properties of a system under broken time-reversal invariance. These ensembles are known to describe the generic fluctuation properties of chaotic quantum systems very accurately [3,5]. We write the $N \times N$ random matrix representing the total Hamiltonian as a sum of a regular and a chaotic part,

$$H(\alpha) = H^{(0)} + \alpha H^{(1)}, \tag{1}$$

where α is the dimensionless transition parameter. The matrices $H^{(1)}$ are drawn from a Gaussian ensemble with the probability density function $P_N^{(1)}(H^{(1)})$. Although the regularity chaos transition is our main interest, we make no assumptions yet for the probability distribution $P_N^{(0)}(H^{(0)})$ of the matrices $H^{(0)}$. The decomposition (1) can be justified for potential and billiard systems. Detailed numerical simulations for the transition of the fluctuations can be found in Ref. [4]. However, despite several attempts, a full-fledged analytical discussion was still lacking. Here, we present a general calculation for the case that $H^{(1)}$ is drawn from the GUE and apply the results for the transition from Poisson regularity to chaos.

As functions of the transition parameter α, we wish to study the k-level correlation functions $R_k(x_1, \ldots, x_k, \alpha)$ depending on k energies x_p, $p = 1, \ldots, k$. To do so, we define the related functions

$$\hat{R}_k(x_1, \ldots, x_k, \alpha) = \frac{1}{\pi^k} \int d[H^{(0)}] P_N^{(0)}(H^{(0)}) \times \int d[H^{(1)}] P_N^{(1)}(H^{(1)}) \prod_{p=1}^{k} \mathrm{tr} \frac{1}{x_p^{\pm} - H(\alpha)}, \tag{2}$$

where the energies are given imaginary increments such that $x_p^{\pm} = x_p \pm i\varepsilon$. The signs are not correlated for different p. The physically interesting functions $R_k(x_1, \ldots, x_k, \alpha)$ are the correlations involving solely the imaginary parts of the Green function. Hence, they are linear combinations of the functions (2). Advantageously, the latter can be written as the derivatives

$$\hat{R}_k(x_1, \ldots, x_k, \alpha) = \frac{1}{(2\pi)^k} \frac{\partial^k}{\prod_{p=1}^{k} \partial J_p} Z_k(x + J, \alpha) \Big|_{J=0} \tag{3}$$

of a normalized generating function $Z_k(x + J, \alpha)$. The energies and the source variables are ordered in the diagonal matrices $x = \mathrm{diag}(x_1, x_1, \ldots, x_k, x_k)$ and $J = \mathrm{diag}(-J_1, +J_1, \ldots, -J_k, +J_k)$, respectively. The physically relevant correlations $R_k(x_1, \ldots, x_k, \alpha)$ are generated by the function $\Im Z_k(x + J, \alpha)$ where the symbol $\Im$ stands for the proper linear combination. Using the standard techniques of the supersymmetry method [7,8] the average over the GUE can be performed directly and the generating function acquires the form

$$Z_k(x + J, \alpha) = \int d[H^{(0)}] P_N^{(0)}(H^{(0)}) \int d[\sigma] Q_k(\sigma, \alpha) \times \mathrm{sdet}^{-1}[(x^{\pm} + J - \sigma) \otimes 1_N - 1_{2k} \otimes H^{(0)}], \tag{4}$$

where σ is a $2k \times 2k$ Hermitian supermatrix. For details of the derivation and notation, the reader is referred to Ref. [9]. We introduced the $N \times N$ and the $2k \times 2k$ unit matrices 1_N and 1_{2k}. The normalized graded probability density function

$$Q_k(\sigma, \alpha) = 2^{k(k-1)} \exp\left(-\frac{1}{\alpha^2} \mathrm{str}\sigma^2\right) \tag{5}$$

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reduces to the well defined superspace δ function [7,9] $\delta(\sigma)$ in the limit $\alpha \to 0$. Thus, for vanishing transition parameter, $Z_k(x + J, \alpha)$ becomes indeed the generating function

$$Z_k^{(0)}(x + J) = \int d[H^{(0)}] P_N^{(0)}(H^{(0)}) \times \mathrm{sdet}^{-1}[(x^{\pm} + J) \otimes 1_N - 1_{2k} \otimes H^{(0)}] \tag{6}$$

of the arbitrary correlations $R_k^{(0)}(x_1, \ldots, x_k)$. We now make the crucial observation that the shift $\sigma \to \sigma - x - J$ removes all angular degrees of freedom in the supermatrix σ from the superdeterminant in Eq. (4). Hence, introducing a $2k \times 2k$ unitary supermatrix u and the diagonalization $\sigma = u^{-1} s u$, we can rewrite the generating function for the transition in the form

$$Z_k(x + J, \alpha) = \int Q_k(\sigma^{\mp} - x - J, \alpha) Z_k^{(0)}(s)\, d[\sigma], \tag{7}$$

where the k eigenvalues $s_{p1}, p = 1, \ldots, k$, in the boson boson and the k eigenvalues $is_{p2}, p = 1, \ldots, k$ in the fermion sector are ordered in the matrix $s = \mathrm{diag}(s_{11}, is_{12}, \ldots, s_{k1}, is_{k2})$.

In order to work out the correlations, we do not use the coset method of Refs. [7,8]; we rather rely on the graded eigenvalue method developed in Refs. [9,10]. It rests on the fact that the average over the unitary supergroup in expressions of the type (7) can be done in one single step. We need the transformation of the Cartesian volume element to eigenvalue-angle coordinates [9] which reads $d[\sigma] = B_k^2(s) d[s] d\mu(u)$ where $d[s]$ is the product of the eigenvalue differentials and $d\mu(u)$ is the invariant Haar measure of the unitary supergroup. The square root of the Jacobian, here referred to as Berezinian, is given by $B_k(s) = \det[1/(s_{p1} - is_{q2})]_{p,q=1,\ldots,k}$ which reflects [9] the determinant structure [6] of the GUE correlation functions. The angular average can be performed by using the supersymmetric generalization [9] of the Harish-Chandra-Itzykson-Zuber integral [11]. We define a second Hermitian supermatrix $v^{-1} r v$ where v is superunitary and r diagonal. The angular average over the shifted Gaussian probability density function yields

$$\int Q_k(u^{-1} s u - v^{-1} r v, \alpha)\, d\mu(u) = \frac{G_k(s, r, \alpha)}{B_k(s) B_k(r)}. \tag{8}$$

For the Gaussian kernel on the right hand side it is sufficient [9,10] to write $G_k(s, r, \alpha) = G_k(s - r, \alpha)$ with

$$G_k(s - r, \alpha) = \frac{1}{\sqrt{\pi \alpha^2}^{2k}} \exp\left[-\frac{1}{\alpha^2} \mathrm{str}(s - r)^2 \right] \tag{9}$$

if further integration over the eigenvalues s is required. If not, all terms involving the permutations of the eigenvalues r_{p1} and ir_{p2} have to be added as in the ordinary case [11]. Furthermore, the integration over s requires one to take a new type of boundary contribution [7,8,12] into account which does not occur in ordinary analysis. However, in Refs. [9,13] it was shown that we do not need to worry about them when calculating correlation functions of the type we are interested in here. After the integration over the unitary supergroup with the help of Eq. (8), we calculate the required derivatives (3) of the generating function exactly as in Ref. [9]. Collecting everything, we find

$$R_k(x_1, \ldots, x_k, \alpha) = \frac{(-1)^k}{\pi^k} \times \int G_k(s - x, \alpha) \Im Z_k^{(0)}(s) B_k(s)\, d[s] \tag{10}$$

for nonzero α. The case $\alpha = 0$ is trivial by construction.

In order to calculate the generic fluctuations, we have to unfold the correlation functions for large level number N by removing the dependence on the level density. We define new energies $\xi_p = x_p/D, p = 1, \ldots, k$, where the mean level spacing D is of the order $1/\sqrt{N}$. The transition parameter α is defined on the original energy scale and has therefore to be unfolded, too. The new, universal transition parameter $\lambda = \alpha/D$ was first introduced by Pandey [14]. The k-level correlation functions on the unfolded scale $X_k(\xi_1, \ldots, \xi_k, \lambda) = \lim_{N\to\infty} D^k R_k(x_1, \ldots, x_k, \alpha)$ are then generic, i.e., translation invariant over the spectrum. It is useful to unfold the integration variables s in Eq. (10) by making the rescaling $s \to s/D$. We arrive at

$$X_k(\xi_1, \ldots, \xi_k, \lambda) = \frac{(-1)^k}{\pi^k} \times \int G_k(s - \xi, \lambda) \Im z_k^{(0)}(s) B_k(s)\, d[s] \tag{11}$$

for nonzero λ where the unfolded generating function of the arbitrary correlations is given by $z_k^{(0)}(s) = \lim_{N\to\infty} Z_k^{(0)}(Ds)$. Hence, we have expressed the unfolded k-level correlation function for the transition from arbitrary to GUE fluctuations as a $2k$-fold integral.

The integral representation (11) immediately implies the translation invariance. Since the arbitrary correlations $X_k^{(0)}(\xi_1, \ldots, \xi_k)$ are assumed to be generic, the generating function $\Im z_k^{(0)}(s)$ can depend only on the differences of the eigenvalues s_{p1} and is_{p2}. Consequently, the shift $s \to s - \xi$ in Eq. (11) makes the right hand side a function of the energy differences $\xi_p - \xi_q$ alone; this is the translation invariance. This observation has two important consequences. First, the level density becomes automatically unity everywhere as it should be. For $k = 1$, the generating function is only a function of $s_{11} - is_{12}$ and, by construction, its power series expansion has to start with $-\pi(s_{11} - is_{12})$. This allows one to do both integrals by applying a standard mean value theorem of complex analysis yielding the value of the integrand at $s_{11} - is_{12} = 0$

and thus $X_1(\xi_1, \lambda) = 1$. Second, the two-level correlation function can be expressed as a double integral. We introduce the sum $\tilde{r} = \xi_1 + \xi_2$ and the difference $r = \xi_1 - \xi_2$ of the energies. According to this rotation in energy space, we make the change $s_1 = s_{11} + s_{21}$, $t_1 = s_{11} - s_{21}$, $s_2 = s_{12} + s_{22}$, and $t_2 = s_{12} - s_{22}$ of the integration variables. Since $\Im z_2^{(0)}(s)$ depends only on the differences of the eigenvalues, it cannot depend on $s_1 + is_2$. Thus, we write $\Im z_2^{(0)}(s) = \Im z_2^{(0)}(s_1 - is_2, t_1, t_2)$ which, for reasons of consistency, should be even in each of the differences t_1 and t_2. By using the aforementioned integral theorem, the integrals over s_1 and s_2 can be performed straightforwardly and the dependence on $\tilde{r}$ disappears. The two-level correlations can thus be cast into the form

$$X_2(r, \lambda) = \frac{8}{\pi^3 \lambda^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\lambda^2}(t_1^2 + t_2^2)\right) \times \sinh \frac{rt_1}{\lambda^2} \sin \frac{rt_2}{\lambda^2} \frac{t_1 t_2}{(t_1^2 + t_2^2)^2} \times \Im z_2^{(0)}(0, t_1, t_2)\, dt_1\, dt_2 \tag{12}$$

which is usually not amenable to further analytical treatment. For the higher correlations with $k > 2$, similar simplifications are likely to exist.

All results derived so far are correct for arbitrary initial correlations $R_k^{(0)}(x_1, \ldots, x_k)$ or $X_k^{(0)}(\xi_1, \ldots, \xi_k)$. We now apply them to the case of a Poisson, i.e., correlation-free, initial spectrum. The Poisson probability density function reads simply

$$P_N^{(0)}(H^{(0)}) = \prod_{n=1}^{N} p^{(0)}(H_{nn}^{(0)}) \prod_{n>m} \delta(\mathrm{Re} H_{nm}^{(0)}) \delta(\mathrm{Im} H_{nm}^{(0)}) \tag{13}$$

where $p^{(0)}(z)$ is a smooth, symmetric, but otherwise arbitrary, normalized probability density function. It is easily shown that there are no correlations and that the level density is given by $R_1^{(0)}(x) = N p^{(0)}(x)$. We find from Eq. (6) for the generating function

$$Z_k^{(0)}(s) = [\zeta_k^{(0)}(s)]^N$$

$$\zeta_k^{(0)}(s) = \int_{-\infty}^{+\infty} p^{(0)}(z) \prod_{p=1}^{k} \frac{is_{p2} - z}{s_{p1}^{\pm} - z}\, dz\,. \tag{14}$$

In order to compute the integral we use the identity

$$\prod_{p=1}^{k} \frac{is_{p2} - z}{s_{p1}^{\pm} - z} = 1 + \sum_{p=1}^{k} \frac{b_p(s)}{s_{p1}^{\pm} - z}$$

$$b_p(s) = (is_{p2} - s_{p1}) \prod_{q \neq p} \frac{is_{q2} - s_{p1}}{s_{q1}^{\pm} - s_{p1}^{\pm}} \tag{15}$$

which can be proven by induction or by standard methods of complex analysis. Hence, the integral in Eq. (14) becomes

$$\zeta_k^{(0)}(s) = 1 + \frac{\pi}{N} \sum_{p=1}^{k} b_p(s) \hat{R}_1^{(0)}(s_{p1})\,, \tag{16}$$

where $\hat{R}_1^{(0)}(x)$ is the Stiltjes transform of $R_1^{(0)}(x)$. To evaluate the generating function on the unfolded scale, it is reasonable to choose the two limiting one point functions equal, we set $\hat{R}_1^{(0)}(x) = \hat{R}_1^{(1)}(x)$. Using $b_p(Ds) = D b_p(s)$ and $\hat{R}_1^{(0)}(D s_{p1}) \to \mp i/D$ for $N \to \infty$, we find

$$z_k^{(0)}(s) = \prod_{p=1}^{k} \exp[\mp i \pi b_p(s)]\,. \tag{17}$$

The signs are determined by the choice of the sign of the imaginary increment in the Green function. The two-level correlation function can be worked out by using the general result (12). In the coordinates introduced above, the initial condition takes the form

$$\Im z_2^{(0)}(0, t_1, t_2) = \frac{1}{2} \mathrm{Re}\left[\exp\left(-i\pi \frac{t_1^2 + t_2^2}{2t_1^-}\right) - 1\right] \tag{18}$$

in which the exponential has to be interpreted as a power series involving the operator $1/t_1^-$. With the help of Ref. [15] we construct the integral representation

$$\Im z_2^{(0)}(0, t_1, t_2) = \frac{\sqrt{2\pi(t_1^2 + t_2^2)}}{4} \int_0^{\infty} I_1\left[\sqrt{2\pi(t_1^2 + t_2^2)\kappa}\right] \frac{\exp(-\varepsilon\kappa)\cos(t_1\kappa)}{\sqrt{\kappa}}\, d\kappa\,, \tag{19}$$

where $I_1(z)$ is the modified Bessel function of first order. This representation allows the evaluation of one of the three integrals left in Eq. (12). Introducing polar coordinates for t_1 and t_2, the angular integral becomes the Bessel function $J_2(z)$ and we arrive at

$$X_2(r, \lambda) = \frac{4}{\pi\lambda^2} \mathrm{Im} \int_0^{\infty} d\kappa \left(\frac{1}{\kappa + i2r/\lambda^2} + \frac{1}{\kappa}\right) \times \int_0^{\infty} d\rho \exp\left(-\frac{\rho^2}{2\lambda^2}\right) \frac{I_1(\sqrt{2\pi\kappa}\,\rho)}{\sqrt{2\pi\kappa}} \times J_2\left(\sqrt{\kappa^2 + i2\kappa r/\lambda^2}\,\rho\right) \tag{20}$$

which states our final result. Since the κ integral converges as it stands if the ρ integration is done first, we have already taken the limit $\varepsilon \to 0$.

The two-level correlation function was already calculated for small values of the transition parameter by means of perturbation expansions in Refs. [16,17]. It can easily be shown that our formula (20) reproduces these results. Lenz [18] derived an integral representation of the function $R_2(x_1, x_2, \alpha)$. Unfortunately, it involves nontrivial ratios of N-dependent determinants which prevented up to now the evaluation of the limit of infinitely many levels. Hence, by using the graded eigenvalue method, we

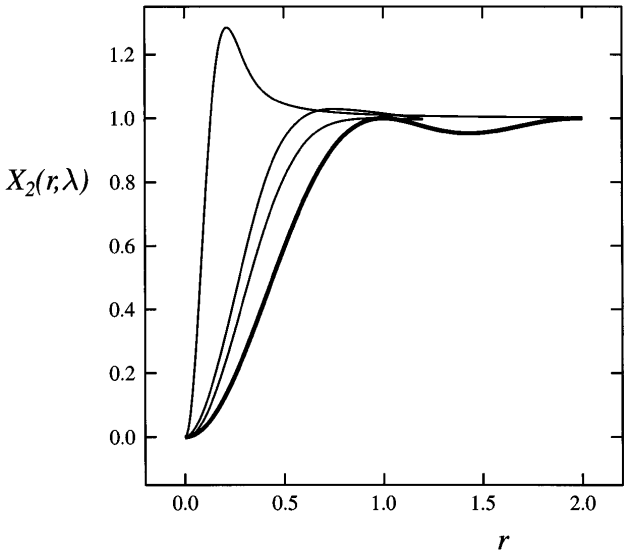

FIG. 1. Two level correlation function of the transition from Poisson to GUE for different values of the transition parameter λ, calculated numerically using formula (20). In the Poisson case, i.e., for $\lambda = 0$, the function is unity which is not drawn. The three thin solid lines correspond from the left to the right to λ values of 0.1, 0.5, and 0.7, respectively. The thick solid line is the pure GUE case corresponding to $\lambda \to \infty$.

present for the first time an expression for the two-point correlation function on the unfolded scale which is valid for all values of the transition parameter λ. Since a further analytical discussion seems impossible, we evaluate formula (20) numerically for λ values of 0.1, 0.5, and 0.7; the results are shown in Fig. 1. The overshoot over unity for small λ values was already observed in Ref. [17]. Obviously, the limiting GUE result is reached rather quickly when λ approaches values of about unity. This confirms the numerical simulations of Ref. [4] and is also comparable to similar transitions.

Formula (20) allows to express the level number variance as a double integral, too. Furthermore, it is conceptually of considerable interest that the transitions discussed here can be formulated as exact diffusive processes in the curved space of the eigenvalues s. Importantly, this can also be done for the GOE and the GSE. These and related issues will be discussed in a forthcoming publication.

In conclusion, we have derived a $2k$-dimensional integral representation for the k-level correlation functions of the transition from arbitrary to chaotic fluctuation properties in a time-reversal noninvariant system. The two-level correlation function could be reduced to a double integral. We have applied these results to the crossover from Poisson regularity to chaos and, for the first time, given a formula that is valid for the entire transition.

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