Skyrmions without Sigma Models in Quantum Hall Ferromagnets

A. H. MacDonald,[1] H. A. Fertig,[2] and Luis Brey[3]
[1]*Department of Physics, Indiana University, Bloomington, Indiana 47405*
[2]*Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506-0055*
[3]*Instituto de Ciencia de Materiales (CSIC), Universidad Autónoma C-12, 28049 Madrid, Spain*
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We report on a microscopic theory of the Skyrmion states that occur in the quantum Hall regime. The theory is based on the identification of Skyrmion states in this system with zero-energy eigenstates of a hard-core model Hamiltonian. We find that for N_ϕ orbital states in a Landau level, a set of Skyrmion states with orbital degeneracy $N_\phi - K$ and spin quantum number $S = N/2 - K$ exists for each non-negative integer K. The energetic ordering of states with different K depends on the interaction potential.

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The ground state of a two-dimensional electron system (2DES) in the strong magnetic field regime of the quantum Hall effect is ferromagnetic at certain values of the Landau level filling factor ν. ($\nu \equiv N/N_\phi$ where N is the number of particles, $N_\phi = AeB/hc \equiv A/2\pi\ell^2$ is the orbital degeneracy of the Landau levels, and ℓ is used below as the unit of length.) The simplest example of a quantum Hall ferromagnet (QHF) occurs at $\nu = 1$, where the ground state is a strong ferromagnet with total spin quantum number $S = N/2$. Phenomena associated with *spontaneous* magnetic order are open to experimental study in QHF's, despite the strong magnetic fields, because the Zeeman coupling to the electronic spins is smaller than other typical energy scales of the system and can even be tuned to zero, for example, by application of hydrostatic pressure to the host semiconductor. QHF's have the unusual property, first identified in finite-size exact diagonalization studies [1] and dramatically evident in recent Knight shift spin-polarization measurements [2], that S can be sharply reduced [3], in appropriate circumstances even to $S = 0$, by the addition or removal of a single electron. This behavior may be explained [4] using a nonlinear σ model (NLσ) continuum field theory description of QHF's. In two dimensions, the NLσ model supports spin texture excitations, known as Skyrmions, that carry a unit quantized topological charge [5]. In QHF's, Skyrmions also carry an electrical charge [4,6,7] equal to the product of the ground state filling factor and the topological charge. This implies that Skyrmions will be present in the ground state for ν close but not equal to 1, explaining [3] the reduction in the spatially averaged moment.

While the NLσ model provides a pleasing qualitative explanation of the spin-polarization experiments, it cannot be used to address quantitative issues. It is valid only for slowly varying spin textures, while the Zeeman coupling at experimentally relevant fields favors small Skyrmion states with only a few reversed spins and relatively rapid variation in the spin-moment orientation. Recent microscopic Hartree-Fock [3,7] estimates of the optimal Skyrmion size agree well with experiment [2], further bolstering evidence for unit charge, large spin, quasiparticles of the $\nu = 1$ ferromagnetic ground state. In this Letter we report on an independent, fully microscopic, picture of QHF Skyrmions. In addition to giving an alternate physical picture of these exotic quasiparticles, our approach has the advantage that we are able to determine precisely the quantum numbers and multiplicities of all Skyrmion states.

The approach we have taken is in the same spirit as the illuminating outlook on the spin-polarized fractional quantum Hall effect that arises from appropriate hard-core model Hamiltonians [8,9]. As discussed for the case of interest below, these models have zero-energy many-particle eigenstates that are often known analytically, are separated from other many-particle states by a finite gap, and have a degeneracy that decreases with increasing N. The incompressible state responsible [10] for a quantum Hall effect transport anomaly in such a model is the nondegenerate maximum N zero-energy eigenstate. The zero-energy eigenstates at lower densities constitute the portion of the spectrum that involves only the degrees of freedom of, in general, the fractionally charged [11] quasiholes of the incompressible state. It is assumed that the difference between the model Hamiltonian and the true Hamiltonian is a sufficiently weak perturbation that the quasihole states are still well separated from other states in the Hilbert space, although accidental degeneracies will be lifted in the spectrum of the true Hamiltonian. Here we apply this approach at $\nu = 1$. Our principal results may be summarized as follows. The zero-energy N-fermion eigenstates for a single hole in a Landau level may be mapped to a set of N-boson states in which the bosons are allowed to occupy only four single-particle states. Single-hole states exist with total spin number $S = N/2 - K$ for each non-negative integer K and in the absence of disorder and Zeeman coupling have degeneracy $g = g_{\rm orb}\, g_{\rm spin}$, where $g_{\rm spin} = 2S + 1$ is the spin multiplicity, and the orbital degeneracy $g_{\rm orb} = N + 1 - K$.

For our analysis we use the symmetric gauge in which the single-particle orbitals [10] in the lowest Landau level are

$$\phi_m(z) = \frac{z^m}{(2^{m+1}\pi m!)^{1/2}} \exp(-|z|^2/4)\,, \qquad (1)$$

where [12] $m = 0, 1, \ldots, N_\phi - 1$, $z = x + iy$, and x and y are the Cartesian components of the two-dimensional coordinate. We study here the hard-core model for which the interaction is [13]

$$V = 4\pi V_0 \sum_{i<j} \delta^{(2)}(\vec{r}_i - \vec{r}_j)\,. \qquad (2)$$

At strong magnetic fields the low-energy Hamiltonian is simply the projection of this interaction onto the lowest Landau level [8]. Many-particle wave functions that are zero-energy eigenstates of this Hamiltonian must vanish when any two particles are at the same position and must therefore have the difference coordinate for each pair of particles as a factor

$$\Psi[z,\chi] = \left[\prod_{i<j}(z_i - z_j)\right]\Psi_B[z,\chi]\,. \qquad (3)$$

We note that the each complex coordinate appears to the power $N - 1$ in the factor in square brackets in Eq. (3) and that this factor is completely antisymmetric. It follows that $\Psi_B[z]$ must be a wave function for N *bosons* and that these bosons can be in states with angular momenta from 0 to $N_\phi - N$. This simple observation leads to the conclusions we reach below.

First, we consider the case of a filled Landau level, $N = N_\phi$. In this case all bosons must be in orbitals with $m = 0$. $\Psi_B[z,\chi]$ must then be proportional to a symmetric many-particle spinor and therefore have total spin quantum number $S = N/2$. The orbital part of the fermion wave function can be recognized as the Slater determinant with all orbitals from $m = 0$ to $m = N_\phi - 1$ occupied. We are able to conclude that the ground state is a strong ferromagnet with no orbital degeneracy. The ease with which this conclusion can be reached contrasts markedly with the case of the Hubbard model where enormous effort has yielded relatively few firm results [14]. When Zeeman coupling is included in the Hamiltonian the ground state will be the member of this multiplet for which all spins are aligned with the magnetic field, i.e., the state with $S_z = S = N/2$.

Our primary interest here is in the elementary charged excitations of the ferromagnetic $\nu = 1$ ground state that occur at $N = N_\phi \pm 1$. For the case of a single *hole* ($N = N_\phi - 1$) the lowest energy states are the zero-energy eigenstates of the hard-core model. We will see that they are the quantized version of the charged Skyrmion states in the NLσ model description of quantum Hall ferromagnets [4]. From Eq. (3) it follows that these states can be mapped to spin-1/2 N-boson states, where the bosons can have angular momentum equal to 0 or 1. In understanding these states it is helpful to start from the state with boson occupaton numbers $n_{1\uparrow} = N$, $n_{1\downarrow} = 0$, $n_{0\uparrow} = 0$, and $n_{0\downarrow} = 0$. Using Eq. (3) we see that the corresponding fermion state has fermion occupation numbers $n_{m\uparrow} = 1$ for $m = 1, \ldots, N_\phi$ and $n_{0\uparrow} = 0$, $n_{m\downarrow} \equiv 0$, i.e., it is the fully polarized state with a single hole in the $m = 0$ orbital. This state is the unique state in the one hole Hilbert space with the maximum possible total boson angular momentum ($M = N$) *and* the maximum S_z ($=N/2$). The set of boson states with angular momentum $M = N - \delta M$ and $S_z = N/2 - \delta S_z$ have boson occupation numbers satisfying

$$\begin{aligned} n_{0\downarrow} + n_{0\uparrow} &= \delta M\,, \\ n_{0\downarrow} + n_{1\downarrow} &= \delta S_z\,. \end{aligned} \qquad (4)$$

For fixed δM and δS_z, the state may be specified by $n_{0\downarrow}$, which can assume values from 0 to the mimimum of δM and δS_z; the number of states is $g(\delta S_z, \delta M) = 1 + \inf[\delta S_z, \delta M]$. We now deduce the total spin quantum numbers of the quasihole states from this expression.

Since the Hamiltonian is spin rotationally invariant, S^2 and S_z must be good quantum numbers, so that all eigenstates occur in spin multiplets with degeneracy $2S + 1$ for total spin S. Furthermore, assuming that edge effects are irrelevant, the Hamiltonian is invariant under simultaneous translation of all coordinates. It follows that δM is also a good quantum number, and that each member of a spin multiplet has associated with it a large orbital degeneracy that scales with the system size. Orbitally degenerate states can be generated from a seed state with minimum δM by repeated application of the operator that lowers the center-of-mass angular momentum [15]. For example, when $S_z = N/2$ this procedure generates holes that occur in successively larger single-particle angular momentum states.

The $S_z = N/2 - 1$ manifold has one state with $\delta M = 0$ and two states for each $\delta M \geq 1$. One state at each δM is the $S_z = N/2 - 1$ member of the $S = N/2$ spin multiplet with the same δM. It follows that there is one $S = N/2 - 1$ spin multiplet with an orbital degeneracy that is reduced by one compared to the $S = N/2$ states. Continuing in the same way, we may conclude that there is a single spin multiplet with $S = N/2 - K$ for each non-negative integer K with orbital degeneracy $N_\phi - K$. Thus the quantized Skyrmion states occur in degenerate manifolds labeled by an integer K and with dimension $(N_\phi - K)(N - 2K + 1)$. The spin degeneracy $(N - 2K + 1)$ is lifted by the Zeeman coupling and is the quantum counterpart of the arbitrary global orientation of a classical Skyrmion. The orbital degeneracy $(N_\phi - K)$ is the quantum counterpart of the arbitrary location of the center of a classical Skyrmion and is lifted by a disorder potential. The density of Skyrmion states in the presence of Zeeman coupling and weak disorder is indicated schematically in Fig. 1.

Repeated application of the total spin lowering operator and the center-of-mass lowering operator allows all states in the $S = N/2 - K$ manifold to be generated from the seed state that has $S_z = S$ and $\delta M = K$. To

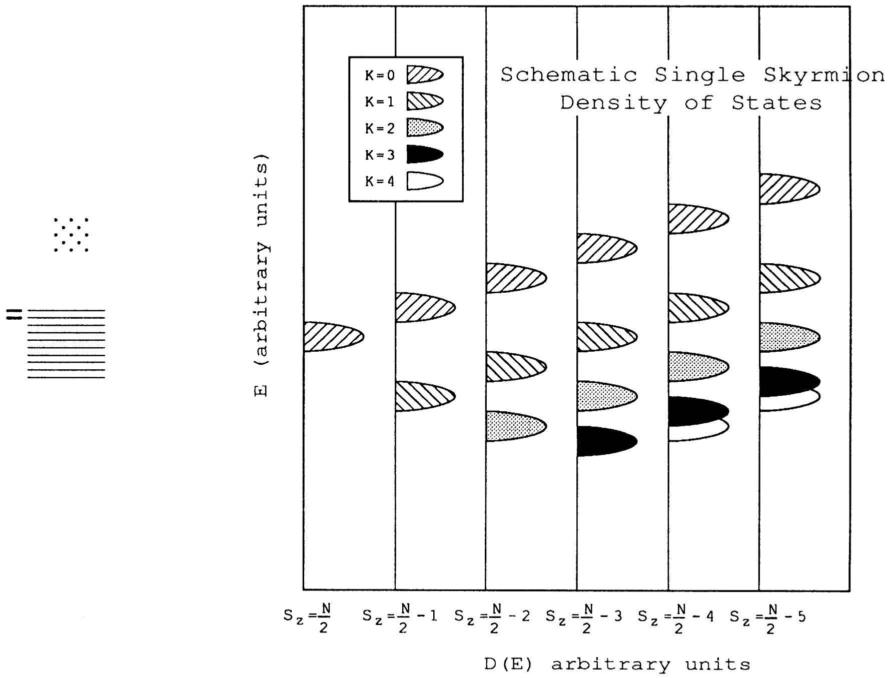

FIG. 1. Schematic single Skyrmion density of states. For quasihole states the orbital degeneracy for each $S = N/2 - K$, $K = 0, 1, 2, \ldots$, and $S_z = -S, \ldots, S$ is lifted by disorder producing a finite width band of states. The spin-multiplet structure persists in the presence of disorder. The energetic offset of bands with the same K and different S_z is due to Zeeman coupling. For the situation illustrated, the Zeeman spin-splitting energy is comparable to the disorder produced band width. The dependence of the $S_z = S$ energy on K depends on the interaction Hamiltonian and the strength of the Zeeman coupling; the situation illustrated where the lowest energy state occurs at $S_z = S = N/2 - 3$, is typical. The $S_z = S = N/2 - 5$ band has been removed from this illustration for clarity.

determine the many-body wave function of this spin-K Skyrmion state, we consider the effect of a total spin raising operator, S_+, on the $K + 1$ boson states that occur at $\delta M = \delta S_z = K$

$$\begin{aligned} S_+|n_{0\downarrow} = k, n_{0\uparrow} = K - k, n_{1\downarrow} = K - k\rangle \\ = \sum_{m=0}^{1} b_{m\uparrow}^{\dagger} b_{m\downarrow} |n_{0\downarrow} = k, n_{0\uparrow} = K - k, n_{1\downarrow} = K - k\rangle \\ \approx \sqrt{(K - k)N}\, |n_{0\downarrow} = k, n_{0\uparrow} = K - k, n_{1\downarrow} = K - k - 1\rangle . \end{aligned} \tag{5}$$

Here $b_m^\dagger$ and b_m are boson creation and annihilation operators. In the last form of Eq. (5) we have included only the $m = 1$ term that dominates for finite K and $N \to \infty$ because $\sqrt{n_{1\uparrow}} \approx \sqrt{N} \gg 1$. In this limit the state with $k = K$ is annihilated by S_+ and is therefore the seed state of the $S = N/2 - K$ Skyrmion multiplet. In first quantization the (unnormalized) boson wave function of this state is

$$\begin{aligned} |\Psi_K^{SK}\rangle = \sum_{i_1,\ldots,i_K}{}' \left[\prod_{j \in \{i_K\}} |\downarrow\rangle_j \exp(-|z_j|^2/4) \right] \\ \times \left[\prod_{l \ni \{i_K\}} |\uparrow\rangle_l z_l \exp(-|z_l|^2/4) \right] . \end{aligned} \tag{6}$$

In these wave functions, the sums are over distinct particle indices, up-spin quasiparticles occupy states with angular momentum $m = 1$, whereas down-spin particles occupy $m = 0$ states; this correlation between spin and angular momentum states is an essential aspect of Skyrmion states [3,7].

We now establish some relationships between the properties of these wave functions and previously known results. As explained above, the $K = 0$ Skyrmion wave function is identical to a Hartree-Fock quasihole. On the other hand, the large K limit of the these wave functions may be related to classical field theory Skyrmions. Consider the fermion wave function

$$\begin{aligned} |\Psi(\lambda)\rangle = \sum_{K=0}^{\infty} \lambda^K |\Psi_K^{SK}\rangle \\ = \prod_{i<j} (z_i - z_j) \prod_l (z_l |\uparrow\rangle_l + \lambda |\downarrow\rangle_l) \\ \times \exp(-|z_l|^2 4) . \end{aligned} \tag{7}$$

The equivalence between the two expressions for $|\Psi(\lambda)\rangle$ is easily established by identifying the coefficient of λ^K in $\prod_l (z_l| \uparrow\rangle_l + \lambda| \downarrow\rangle_l)$ For large λ, the sum in Eq. (7) will be dominated by terms in a relatively narrow range of K values [16]. $|\Psi(\lambda)\rangle$ is precisely the single Slater determinant proposed by Moon *et al.* [7] as a microscopic trial wave function for the Skyrmion on the grounds that, for large λ, it gives a spin texture in precise agreement with that of the classical Skyrmion [4] of size λ.

The more general Hartree-Fock (HF) single-Slater-determinant wave functions of Ref. [3] may be written in the form

$$|\Psi^{\mathrm{HF}}\rangle = \prod_m (u_m c_{m\downarrow}^\dagger + v_m c_{m+1\uparrow}^\dagger)|0\rangle\,. \qquad (8)$$

The HF approximation is equivalent to minimizing the energy of this wave function subject to a normalization constraint. An obvious deficiency of the Hartree-Fock approximation is its failure to reflect known symmetries of the Hamiltonian. In particular, the Hartree-Fock wave function is not an eigenstate of S_z or M. As discussed by Nayak and Wilczek [17], this failure is readily remedied by projecting the Hartree-Fock state onto a state of definite S_z (and, therefore, definite M). For the hard-core model, the HF equations [3] may be solved *analytically*, with the result $|u_m|^2 = 1 - |v_m|^2 = \lambda^2/[\lambda^2 + 2(m + 1)]$, where λ is a free parameter. It is easily verified that the corresponding wave function is precisely $|\Psi(\lambda)\rangle$. In the case of the hard-core model the projection of the Hartree-Fock wave function onto a state of definite S_z yields the exact Skyrmion wave functions, suggesting that this seemingly *ad hoc* procedure might be generically accurate.

We remark that all the results we have obtained here to a hole in the ferromagnetic ground state apply equally well to the case of a particle added to the ferromagnetic ground state because of an exact particle-hole symmetry [18] that holds near $\nu = 1$. For the case of the hard-core model the energy of the particle states is $4V_0$, independent of K. Our analysis is also readily generalized for fractional filling factors $\nu = 1/m$, although in that case the quasiparticle states cannot be generated by particle-hole transformation. For general interactions the Skyrmion energy is dependent on K; the minimum energy state may occur at $K = 0$ where Hartree-Fock theory is valid, at $K \to \infty$ where the classical field theory becomes valid, or at an intermediate value of K where explicit expressions for the energy are not available. The many-particle wave functions that we have derived for the Skyrmion states are exact only for the hard-core model. The accuracy of these wave functions for general interactions, which has been confirmed by exact diagonaliztion [19], rests on the existence of a gap for charged excitations, and they are, in this sense, analogous to Laughlin's trial wave functions for incompressible states at fractional filling factors.

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