Formation of Avalanches and Critical Exponents in an Abelian Sandpile Model

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The structure of avalanches in the Abelian sandpile model on a square lattice is analyzed. It is shown that an avalanche can be considered as a sequence of waves of decreasing sizes. Being more simple objects, waves admit a representation in terms of spanning trees covering the lattice sites. The difference in sizes of subsequent waves follows a power law with the exponent α simply related to the basic exponent τ of the sandpile model. Using known exponents for the spanning trees, we derive from scaling arguments $\alpha = 3/4$ and $\tau = 5/4$.

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The sandpile model was introduced in the work [1] by Bak, Tang, and Wiesenfeld to manifest the nature of "self-organized criticality" (SOC). The Abelian version of the model became most popular because it turned out to be analytically tractable [2]. Several characteristics of the Abelian sandpile were evaluated exactly: the total number of allowed configurations in the SOC state [2], the fractional number of sites having a given height [3,4], some height-height correlation functions [3,5], and the expected number of topplings at a given site due to a particle added at another one [2].

Nevertheless, exact values of exponents characterizing avalanche processes remained unknown. The distribution of avalanches obeys the power law $P(S) \sim S^{-\tau}$ in which S is the number of distinct sites toppled during the relaxation. Exponents corresponding to the mass and linear extent of avalanches can be expressed in terms of τ [6,7]. Initial simulation studies of sandpiles [1] gave $\tau = 1$. The first theoretical predictions based on a continuous-energy model [8] and a Flory-like approximation [9] justified this result. Later on, Manna [10] undertook large-scale simulations and obtained the value $\tau = 1.22$. Meanwhile, the data of the majority of numerical experiments were roughly consistent with $\tau = 7/6$ [6]. Simple mean-field arguments by Christensen and Olami [7] led to a somewhat smaller value $\tau = 23/21$.

Recently, Pietronero, Vespignany, and Zapperi [11] presented a renormalization scheme of a new type that allowed them to estimate critical exponents of the sandpile model. They obtained $\tau = 1.253$.

Determination of τ needs a detailed analysis of the relaxation process. It would be desirable to represent the whole avalanche as a series of more elementary events and to express τ via auxiliary exponents. The first step in this direction has been made by Dhar and Manna who introduced the notion of inverse avalanches [12]. It was soon shown that there exists a direct procedure leading to the same representation of avalanches [13]. New objects, being basic elements of the avalanche, were termed "waves of topplings."

In this Letter, we use the wave construction for finding the critical exponents of the 2D Abelian sandpile model. We will show that a typical avalanche can be considered as a sequence of waves of decreasing sizes. Each site evolved into a wave topples only once. This permits us to define a spanning tree representation for waves and to find their distribution exactly. The difference in sizes of subsequent waves Δs also follows the power law $\Delta s \sim s^{\alpha}$, where s is the size of the wave and the exponent α is simply related to τ. The problem of evaluation α can be formulated in terms of spanning trees, or, equivalently, of the q-component Potts model in the limit $q \rightarrow 0$. Using known exponents of the latter model, we will derive the exponent α from scaling arguments. We estimate α from simulations and find good agreement between the measured and derived values.

The model we consider is a cellular automaton defined on a $N \times N$ square lattice $\mathcal{L}$. The sandpile is characterized by the number of particles or integer heights z_i at all sites i and is specified by two rules. (i) Adding a particle at a random site: $z_i \rightarrow z_i + 1$. (ii) Toppling of unstable sites: if any $z_i > 4$, then $z_j \rightarrow z_j - \Delta_{ij}$ for all $j \in \mathcal{L}$.

The toppling matrix Δ is the discrete Laplacian which has, in the case of a square lattice, nonzero elements $\Delta_{ii} = 4$ for all i and $\Delta_{ij} = -1$ for all pairs of adjacent sites i and j. It is convenient to introduce an additional site i_0 connected with all boundary sites to be a sink of toppled particles.

All stable configurations of heights which are allowed in the SOC state have the same probability [2]. To determine if a given configuration is allowed, Majumdar and Dhar [6] have introduced a "toppling from the sink" together with a given order of preference for successive topplings of sites. Using this procedure, one adds a particle to each site connected with i_0. All sites of $\mathcal{L}$ topple exactly once if and only if the configuration is allowed. Drawing all bonds connecting pairs of sites toppled at successive moments of time, one obtains a spanning tree covering a given lattice. The point i_0 is the root of the tree T_0. The collection of all possible rooted spanning trees $\{T_0\}$ is in one-to-one correspondence with the set of allowed configurations.

An avalanche is a perturbation of a stable state. It begins when a particle is dropped on a site of height 4 and stops when all sites become stable again. The Abelian property admits an arbitrary order of topplings of nonstable sites during an avalanche. To introduce the waves of topplings, we carry out the process of relaxation in a specific way [13]. As usual, let us start with adding a particle to the site i of height 4 in an allowed configuration C. Topple it once and then topple all sites that become unstable, keeping the site i out of the second toppling. We call the set of toppled sites "the first wave of topplings."

The site i loses 4 and receives m particles ($0 \leq m \leq 4$) besides the added one during the first wave. If the resulting height $z_i = 5$, we topple the site i a second time and continue the avalanche, not permitting this site to topple a third time. The set of relaxed sites at this stage is "the second wave." The process continues producing intermediate configurations $C_1, C_2, \ldots, C_n$ until the site i becomes stable and the avalanche stops.

All sites involved in the kth wave ($k \geq 1$) topple only once during this wave. Indeed, to topple a site j twice, we have to first topple one of its neighbor sites j_1. The second toppling at j_1 is possible only after the second toppling at its neighbor j_2, $j_2 \neq j_1$ and $j_2 \neq j$. Continuing, we obtain the chain $j_1, j_2, \ldots$, which contains an initial site i for the finiteness of the wave. However, by definition, the site i topples once during the given wave, therefore other sites of the wave topple once as well.

The construction of waves admits a spanning-tree interpretation. For this purpose, we introduce the sandpile model on an auxiliary lattice $\mathcal{L}'$, consisting of the original lattice $\mathcal{L}$, the site i_0, connected with boundary sites of $\mathcal{L}$ and an additional bond connecting the site i_0 and a given site i inside the lattice. If we consider the toppling from the sink for each allowed configuration on the new lattice $\mathcal{L}'$, we obtain, as a result, the set of spanning trees covering $\mathcal{L}'$ and having a root i_0. The trees obtained are of two classes. The first one consists of trees without a bond $(i_0\ i)$ and therefore coincides with the set of one-rooted spanning trees $\{T_0\}$ defined above. The trees of the second class contain the bond $(i_0\ i)$. On removing the bond $(i_0\ i)$ a subtree of the whole tree gets disconnected. We obtain a two-rooted spanning tree on the original lattice $\mathcal{L}$ consisting of two components T_i' and T_0' having the roots at the sites i and i_0.

Now, we can select a particle dropped on i among all particles added to sites connected with i_0. This particle can be considered as a perturbation giving rise to an avalanche on $\mathcal{L}$. Since the site i on a lattice $\mathcal{L}'$ is connected with i_0, it topples only once and this avalanche is actually the wave. The corresponding subtree T_i' and its supplementary component T_0' are the graphic portrait of an intermediate configuration appearing after a given wave. To construct the subtree corresponding exactly to the first wave, one can start with a configuration C which is allowed simultaneously on the lattices $\mathcal{L}$ and $\mathcal{L}'$. To select the kth wave for an arbitrary k, one can first add $k-1$ particles at i and then apply the toppling from the sink. An allowed configuration on $\mathcal{L}$ appears again after the last wave.

The graph representation of waves enables us to link the toppling process with the lattice Green function $G = \Delta^{-1}$, that is, the solution of the Poisson equation with the boundary conditions $G_{i_0 j} = 0$ for all $j \in \mathcal{L}$. In [13] the following proposition has been proven: For a lattice $\mathcal{L}$ with an additional vertex i_0,

$$G_{ij} = \mathcal{N}^{(i,j)}/\mathcal{N}\,, \tag{1}$$

where $\mathcal{N}^{(i,j)}$ is the number of two-rooted spanning trees having the roots i_0 and j such that both the vertices i and j belong to the same subtree; $\mathcal{N}$ is the total number of spanning trees on $\mathcal{L}$.

The wave distribution follows immediately from Eq. (1) and the known asymptotics of the Green function $G(r) \sim \ln r$. Indeed, the relative number of waves $\mathcal{N}(r_w \geq r_{ij})$ whose characteristic radius r_w is not less than the distance between i and j is

$$\mathcal{N}(r_w \geq r_{ij}) \sim G_{ij}\,. \tag{2}$$

Since the waves are compact, their sizes scale as $s \sim r^2$. Then, the asymptotic distribution of sizes $D(s)$ is

$$D(s) \sim P(r)\,\frac{dr}{ds} \sim \frac{1}{s}\,, \tag{3}$$

where $P(r) = dG(r)/dr \sim 1/r$.

The onefold toppling of all sites in a wave is equivalent to a pass of particles over the boundary of the wave from sites inside the wave to neighboring sites outside. Typically, this leads to squeezing the next wave with respect to the previous one because a portion of the sites losing particles becomes unable to topple the next time. So, the subsequent waves $W_1, W_2, \ldots, W_n$ belonging to the same avalanche are generally of decreasing sizes $s_1, s_2, \ldots, s_n$. An avalanche stops just at the moment when the boundary of the last wave reaches the initial point i.

Self-similarity of avalanches implies self-similarity of their components. Therefore, one can expect that the size difference between successive waves $\Delta s = s_k - s_{k+1}$ obeys also a power law

$$\Delta s \sim s^{\alpha}. \tag{4}$$

The exponent α, if it exists, can be related with τ by a scaling relation. Let n denote the number of waves in an avalanche, which coincides with the number of topplings at the site i. Equation (4) can be rewritten in the differential form $ds/dn \sim s^{\alpha}$ or

$$dn \sim \frac{1}{s^{\alpha}}\,ds\,. \tag{5}$$

The wave of size s belongs to an avalanche of size $S \geq s$ which has the probability $P(S \geq s) \sim s^{1-\tau}$. Then, the distribution of waves belonging to diverse avalanches is

$$D(s) \sim \frac{1}{s^{\alpha+\tau-1}}. \quad (6)$$

Comparing (6) with (3), we obtain the scaling relation

$$\alpha + \tau = 2. \quad (7)$$

Majumdar and Dhar [6] introduced an exponent y assuming that n scales with the size of an avalanche as $n \sim s^{y/2}$. To be consistent, the exponents α and y must be related as

$$2\alpha + y = 2. \quad (8)$$

We have studied the statistics of waves numerically generating 10^6 avalanches on the lattices of sizes up to $N = 500$. In Fig. 1, we have plotted Δs vs the wave size s on a log-log scale, which displays a clear power-law behavior.

In [14], Grassberger and Manna introduced clusters of sites A_n which toppled $\geq n$ times, $n \geq 1$, during an avalanche. If waves of a given avalanche obey the relations $W_1 \supset W_2 \supset \cdots \supset W_n$ strictly, the structure of waves coincides completely with that of clusters $\{A\}$. At the same time, Dhar and Manna who investigated inverse avalanches recorded situations when the wave W_k overlaps the preceding one, W_{k-1}. They argued that these events are nevertheless relatively rare, and on the average the last waves scale as the clusters of maximal topplings. Our simulations show generally that the distributions of waves $\{W\}$ and clusters $\{A\}$ follow the same asymptotical law (4). Taking into account these observations, we neglect the overlapping of waves and deal only with the decrease of wave sizes.

The above construction allows us to determine α from scaling arguments. To this end, we have to link the decrease in the size of waves Δs with the spanning-tree characteristics. Given a rooted tree T_i and two sites $j_1, j_2 \in T_i$, we shall say that the site j_1 is a predecessor of j_2 if the unique path connecting j_2 and the root i passes via j_1. It follows from this definition that the site j_2 topples before j_1 during the toppling process.

Let $T_i(W_k)$ be the subtree with a root i corresponding to the wave W_k. As all sites involved in W_k topple exactly once, all internal sites of W_k remain unchanged. The wave W_{k+1} following W_k will repeat its order of topplings until the relaxation process reaches the boundary of W_k. Accordingly, the subtree $T_i(W_{k+1})$ that represents W_{k+1} will coincide with $T_i(W_k)$ as long as its sites have no predecessors among the boundary sites of W_k. Denote by B_j a set of sites of $T_i(W_k)$ having a boundary site j as a predecessor. Actually, B_j is a branch of $T_i(W_k)$ attached to the subtree at the point j. If the site j becomes stable with respect to the next wave W_{k+1}, all sites of B_j become stable too as the toppling process penetrates into B_j via the point j. As a result, the sites of B_j, as well as the site j itself, contribute to Δs. Generally, Δs consists of all boundary sites $j_1, j_2, \ldots$ of the wave W_k becoming stable with respect to W_{k+1} and of sites of all sets $B_{j_1}, B_{j_2}, \ldots$ having $j_1, j_2, \ldots$ as predecessors.

In Fig. 2, we show a typical form of the set contributing to Δs. The external contour Γ represents the boundary sites of the wave W_k, and the loops γ_i correspond to the sets B_i. By construction, the two main quantities to determine Δs are the length of the contour Γ and the area of loops $\{\gamma\}$.

Denoting by R a linear extent of the wave W_k, we can estimate the length of the contour Γ as $R^{5/4}$ since Γ is a *chemical path* on the dual spanning tree [15]. Then, the contribution from Γ gives

$$\Delta s \sim R^{5/4} \sim s^{5/8}, \quad (9)$$

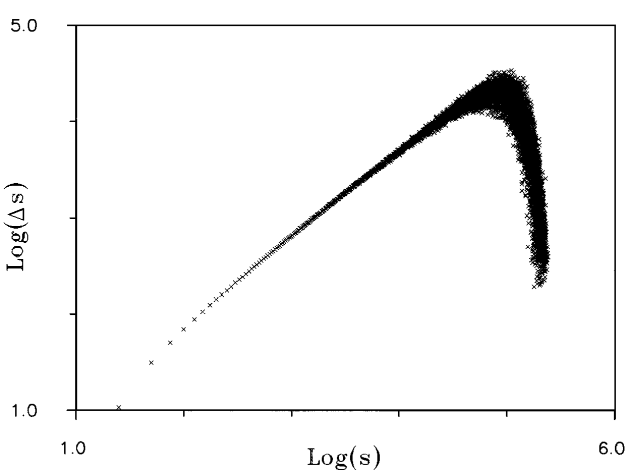

FIG. 1. Double logarithmic plot of averaged decrement Δs against cluster size s for the statistics of 10^6 avalanches on a square lattice of size $L = 500$.

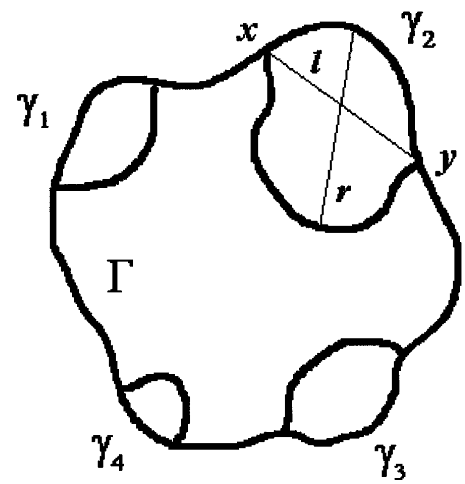

FIG. 2. A typical contour Γ with a set of loops $\{\gamma\}$. The loop γ_2 is attached to Γ in points x and y separated by a distance l. The linear extent of γ_2 is r.

which implies $5/8$ for the exponent α. We shall see, however, that the leading contribution comes from the second quantity determined by the interior of loops $\{\gamma\}$.

Consider a single loop γ. It is characterized by a distance l between points x and y where it is attached to the contour Γ and the linear extent r (see Fig. 2). The cluster surrounded by γ is a subtree having a fixed root at one of the two boundary sites, say, x. According to (3), the trees of linear extent r are distributed as $D(r) \sim 1/r$. The root can occupy any of r^2 positions inside γ. Therefore, subtrees with a fixed root are distributed as $1/r^3$. Let us consider a circle C of radius l having a center at point x. The average number of intersections between C and Γ is of order $l^{1/4}$ due to fractal dimensions of the chemical path. The point y can occupy any of l points of C with equal probability. Thus, we obtain the asymptotical joint distribution of loops γ

$$D_\gamma(l, r) \sim \frac{l^{1/4}}{r^3 l}. \quad (10)$$

The maximal extent of both r and l is of order R. The minimal extent of r is of order l, whereas l is bounded from below by the lattice spacing. Integrating over r and l, we obtain the contribution to Δs from the single loop γ

$$\Delta_\gamma s \sim \int_1^R \int_l^R r^2 D_\gamma(l, r) dr\, dl \sim R^{1/4}. \quad (11)$$

The number of loops is proportional to the length of Γ, that is, $R^{5/4}$. Then, the total Δs is

$$\Delta s \sim R^{3/2} \sim s^{3/4}. \quad (12)$$

Comparing (12) with (4) and using (7), we finally get $\alpha = 3/4$ and $\tau = 5/4$.

Our numerical estimation of $\alpha = 0.73$ extrapolated to infinite N is quite consistent with the obtained value.

The distribution (10) is based on scaling arguments. To verify its validity, we have used an exact result coming from the analogy between a Coulomb gas and spanning trees. Saleur and Duplantier [16] evaluated the probability that vicinities of two points x and y separated by a distance l are connected by two paths on the tree. They found for large l

$$D_2(l) \sim \frac{1}{l^{3/2}}. \quad (13)$$

To derive (13) from (10), we consider two paths as a loop and compare conditions leading to (10) and (13). The distribution (10) is restricted by the presence of the external contour Γ that fixes the position of the initial point x. In the latter case, the point x can occupy any site of the perimeter proportional to $r^{5/4}$. The linear extent r of the loop varies from l to infinity, so the integration over r gives

$$D_2(l) \sim \int_l^\infty r^{5/4} D_\gamma(l, r) dr \sim \frac{1}{l^{3/2}} \quad (14)$$

in accordance with (13).

If α is known, other exponents of the sandpile model can be readily found. For instance, using the identity [6]

$$\tau_s - 1 = 2(\tau - 1)/(2 + y), \quad (15)$$

we find from (7) and (8) the exponent of the total number of topplings $\tau_s = 6/5$.

The numerical result by Manna for $\tau_s = 1.2008$ [17] is in excellent agreement with our theoretical prediction.

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