Identification of Intermittent Ordered Patterns as Heteroclinic Connections

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Cellular flames stabilized on a porous plug burner at low pressure form ordered patterns consisting of concentric rings of cells. In certain regions of the parameter space of propane-air mixtures, ordered patterns appear intermittently. In these states, patterns of concentric rings persist for varying lengths of time, abruptly dissolve into highly irregular structures with no discernible ring pattern, and then reappear as ordered patterns with the same or slightly different numbers of cells. Experimental evidence and theoretical arguments show that these intermittently ordered states correspond to a network of heteroclinic connections among unstable equilibria.

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The phenomenon of intermittency plays a prominent role in deterministic dynamics [1–4]. Pomeau and Manneville were among the first to demonstrate that intermittent dynamics could arise in time series generated from one-dimensional maps or from the solutions of coupled sets of ordinary differential equations. There have been a number of experimental observations of such dynamics [5,6]. In this paper we report the observation of intermittency in a pattern-forming system in which concentric rings of propane-air cellular flames appear and disappear at irregular intervals. We present both quantitative and qualitative evidence to link these intermittently ordered states to heteroclinic connections.

In intermittent regimes of this sort, the trajectory representing the state of the system first approaches the neighborhood of an equilibrium associated with a steady pattern, and then leaves it rapidly as a linearly unstable mode for that pattern grows. The trajectory may then return to the same equilibrium or possibly approach another unstable equilibrium pattern. This process is repeated, so that the solution spends varying periods of time hesitating near unstable equilibria between which excursions (represented as spatial and temporal disorder in the overall pattern) occur. The resulting motion is intermittent, with well-defined events occurring at irregular intervals. The emergence of this type of intermittent state is characteristic of the way in which some spatially extended systems undergo transitions between two ordered states of the system. The symmetry of the burner and the small number of cells play a crucial role in our ability to observe this type of intermittent behavior.

Premixed flames [7] are stabilized on a 5.62 cm circular porous plug burner which is housed in a combustion chamber made from process glass pipe. The working pressure, $\frac{1}{2}$ atm, and the flow rate are controlled to 0.1%. The control parameters are the total flow rate of the gases and the equivalence ratio (the ratio of fuel to oxidizer relative to that at stoichiometry). The premixed propane-air flame is viewed from above by a silicon intensified target camera through a mirror mounted on the top of the chamber. The motion is recorded on videotape with an embedded time code.

As the control parameters are adjusted, the steady uniform flame front breaks up into (brighter, hotter) cells whose boundaries are demarked by (darker, cooler) cusps and folds which curve away from the burner surface. The cells organize into ordered states formed by two concentric rings of cells [8]. These ordered states are labeled according to the number of cells in each ring (e.g., 9/3 designates an ordered state with 9 cells in the outer ring and 3 cells in the inner ring).

In propane-air cellular flames the parameter range between consecutive ordered states contains intermittently ordered states in which concentric rings of cells appear at irregular intervals. In the example presented here a 9/3 ordered state is stable over a range of flow rates. As the flow rate is increased beyond a critical value, this 9/3 state becomes unstable and is replaced by an intermittently ordered state. In the intermittent state, the 9/3 pattern appears for a time, mostly for short times ($\approx 1/3$ sec) but occasionally for long times (120 sec), after which it dissolves into a very complicated spatiotemporal sequence, finally resolving itself into another (or possibly the same) quasistationary pattern.

A frame-by-frame analysis was performed of a two-hour run taken at a flow rate of 8.27 liters/min, an equivalence ratio of 1.85, and a pressure of $\frac{1}{2}$ atm. The flow rate is just beyond the value at which the 9/3 ordered state is stable. Figure 1 shows the four principal patterns of concentric rings (9/3, 9/4, 10/3, and 10/4)

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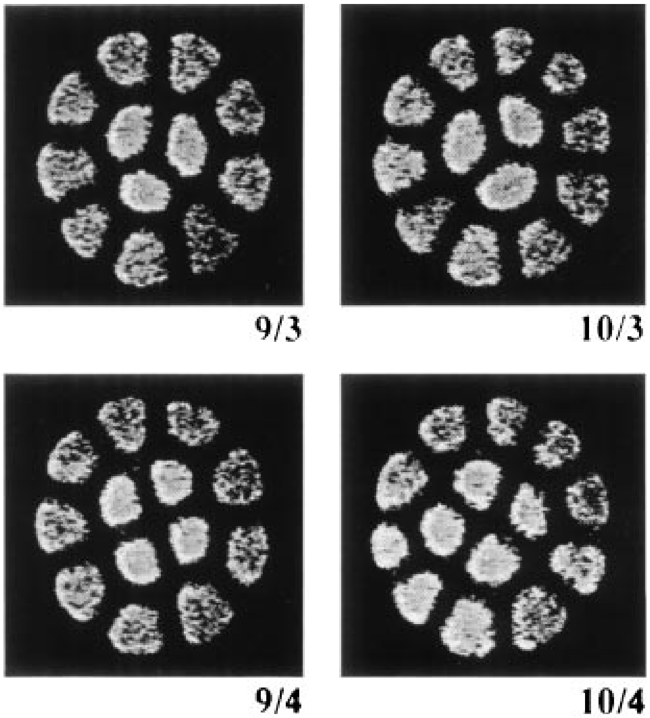

FIG. 1. Four ordered patterns of concentric rings—9/3, 9/4, 10/3, and 10/4—which appear intermittently in the state described in this paper.

which appear at various times. The analysis reveals that the system spends 59% of its total time in excursion between ordered patterns. The 9/3 pattern receives 338 of the visits, 86% of the total, and is the only pattern for which we have sufficiently good statistics to make a quantitative comparison with theory. The other states—9/4, 10/3, and 10/4—receive 9%, 4%, and 1% of the visits, respectively.

Figure 2 shows a passage near an equilibrium. Frames from the videotape of the experiment are numbered to correspond with a schematic diagram of an orbit in a traverse near a saddle point in phase space shown in Fig. 3. As the system moves along the unstable manifold (frames 1–3), the flame front has a highly irregular shape with no discernible structure. As the system passes near the saddle point (frames 4–6), the pattern of concentric rings becomes visually discernible. As the traverse continues away from the saddle point (frames 7–9), the flame resumes its irregular motion.

A qualitative argument that the intermittently ordered states correspond to heteroclinic connections is provided by the observed sequence of transitions. At low values of the flow rate the 9/3 state is stable; but, at a larger value, it becomes unstable to a cycle in which the ordered pattern repeatedly breaks up and reforms. At successively larger values of the flow rate, a 9/4 ordered state becomes stable. The 9/4 state is in turn destabilized at a higher value of the flow rate to a situation in which the 10/4 cell

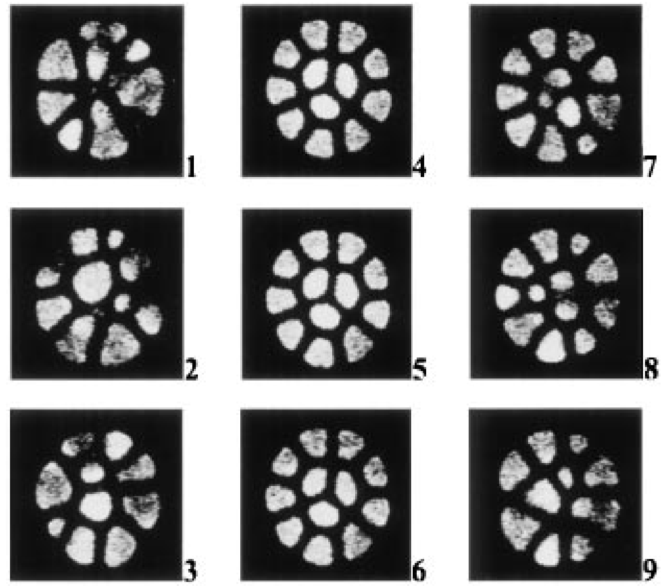

FIG. 2. Nine sequential frames of videotape depicting the passage of the system near an unstable equilibrium (9/3).

state breaks up and reforms. This sequence of transitions suggests global bifurcations in which cycles are formed and broken as the unstable and stable manifolds of the saddle equilibria associated with the stationary states intersect tangentially and then separate. Such a sequence is typical for propane-air cellular flames. Other fuels have bifurcation sequences with different kinds of transitions. In isobutane-air and butane-air flames the predominant type of transition involves ordered states which bifurcate

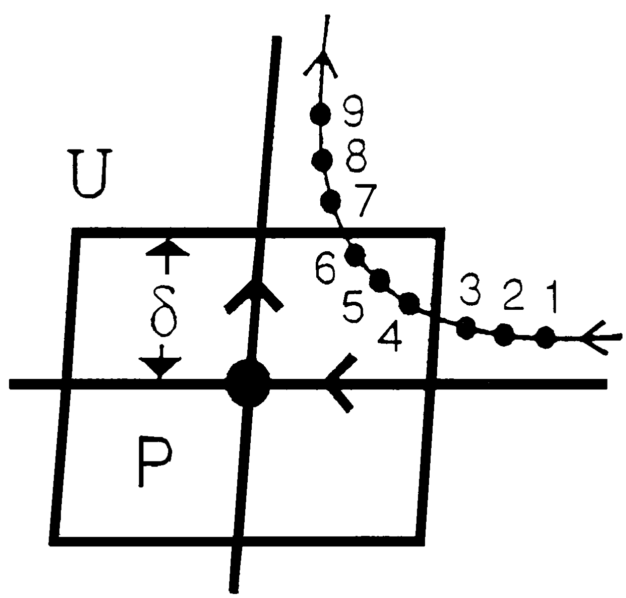

FIG. 3. A schematic diagram of the motion in phase space corresponding to a passage. The numbers correspond to frame numbers in Fig. 2.

directly into other ordered states with different numbers of cells or to dynamic states in which cells in a ring move collectively.

Quantitative evidence of heteroclinic cycles comes from an analysis of the time the system spends hesitating near unstable equilibria. In general, homoclinic and heteroclinic orbits are structurally unstable, meaning that infinitesimally small changes in the parameters can remove the connection. In experimental systems this condition means that such orbits would be not observed, since it is impossible to tune the parameters to the exact value. It has been found, however, that the presence of symmetry groups under which the vector field is equivariant can result in structurally stable orbits [9].

Symmetry-induced heteroclinic cycles are not generally associated with chaotic behavior. Under suitable conditions on the eigenvalues of the saddle point [10], the cycles are asymptotically as well as structurally stable; all orbits nearby approach them as $t \to \infty$. Solutions spend increasing periods near the equilibria, and the heteroclinic excursions become less frequent. There is no inherent time scale in the system, and the period of the cycle approaches infinity.

Busse [11] first noticed the importance of small amplitude perturbations due to round-off errors in his theoretical study of convection patterns in which a set of three ordinary differential equations invariant under cyclic permutations was shown to possess a heteroclinic cycle. Such perturbations may be too small to significantly change the behavior of the system of a heteroclinic excursion because the vector field is large compared to the perturbation. Near an equilibrium where the field becomes vanishingly small, the perturbations can keep an orbit from approaching the equilibrium arbitrarily closely. The time spent by an orbit near a fixed point is limited, and a "statistical limit cycle" is created from the heteroclinic cycle [11]. The period of the cycle is finite and random. In the combustion experiment described in this paper, a source of perturbations is the low-amplitude irregular vibrations characteristic of the ordered states [8]. The switching between roll patterns with different orientations observed by Ning and Ecke [12] in Rayleigh-Bénard convection may be another example of symmetry-induced heteroclinic connections.

The dependence of the time scale of a statistical limit cycle on the largest unstable eigenvalue can be worked out explicitly. In the interest of brevity we refer the reader to [13,14] for more details on the derivation and reproduce the main points only. Consider a point q in the phase space of the governing equations for the physical system. The point q is contained in a heteroclinic orbit to a pair of fixed points $(p+, p-)$ if the orbit $\phi(t)$ based at q approaches $p+$ as $t \to +\infty$ and $p-$ as $t \to -\infty$. The fixed points $p+$ and $p-$ must be saddle points with eigenvalues on both sides of the imaginary axis, hence the term saddle connection. The orbit ϕ is called homoclinic if $p-$ and $p+$ are the same point.

Referring to Fig. 3, the circulation time for trajectories near the cycle will be dominated by the slow passage near the saddle point p, i.e., the time spent in U. We approximate the total circulation time by the sum of a constant "cycle time," and time spent in U, or "passage time" T. It is within U that external perturbations are assumed to have a major effect, so the variation in circulation time is determined by T alone. The calculation can be done by a linear analysis at p, incorporating added noise. Given an incoming distribution of initial data to U, its evolution can be computed and the distribution of passage times through U can be calculated:

$$P(T) = 2\lambda\Delta(T)e^{-\Delta^2(T)}/\sqrt{\pi}(1 - e^{-2\lambda T}), \qquad (1)$$

where $\Delta(t) = \delta[(\epsilon^2/\lambda)(e^{2\lambda t} - 1)]^{-1/2}$, λ is the unstable eigenvalue of p, ϵ is the rms noise level, and δ is the size of the neighborhood in which noise is assumed to have an appreciable effect.

$P(T)$ is a skewed distribution, with its peak off the mean, and it can be shown [15] that it possesses an exponential tail, namely, $P(T) \approx \frac{2\delta}{\sqrt{\pi}\epsilon}\lambda^{3/2}e^{-\lambda T}$ as $T \to \infty$. In [13] it is shown that the derivation can be carried out for systems of higher dimension than two, and in [14] that the details of the perturbing signal are not critical in this derivation.

This theoretical result suggests an experimental technique for analyzing heteroclinic connections. If the time the system spends in a single quasistationary state is recorded for successive visits near that equilibrium, a histogram of these times should coincide with the passage-time distribution derived above. The principal experimental difficulty is the identification of the entrance of the trajectory into the "box" U, shown schematically in Fig. 3. We use the visual criterion that the system point is in the box U around an equilibrium if a pattern of two complete concentric rings of cells can be distinguished. The uncertainty in the experimental identification of the entrance and exit from the box U is ± 2 frames.

The passage times are determined from a frame-by-frame analysis of the videotape. The histogram of the passage times for the $9/3$ pattern in shown in Fig. 4. We first fit the tail of the histogram to a single exponential using times longer than 20 frames. The resulting fit is shown in the inset of Fig. 4. The fitted value 70 ± 2 is the inverse of the unstable eigenvalue. It changes very little (± 1) as the bin size is varied from 20 to 30 frames. The quality of the fit is excellent for this kind of experimental data. The eigenvalue $(1/70)$ associated with the $9/3$ pattern increases smoothly as the control parameters are varied away from the transition point where the pattern first becomes unstable.

The main plot in Fig. 4 shows the full histogram of passage times. The bin size of the three frames has been chosen to reveal the structure of the peak at 8 ± 3 frames. The position of the peak varies slightly as the bin size is changed from two to seven frames. The analytic result for the distribution of passage times $P(T)$

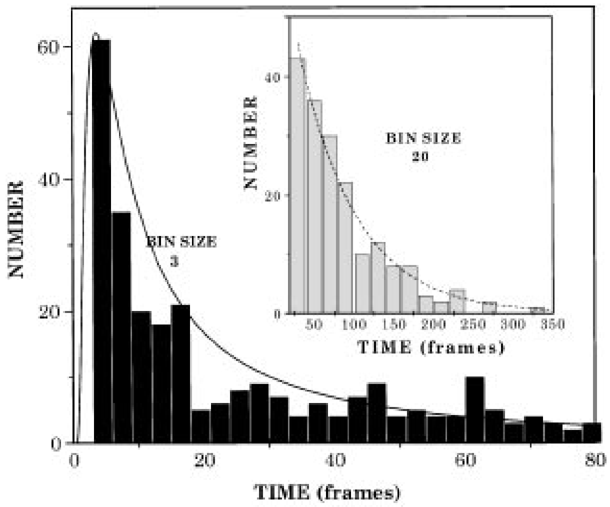

FIG. 4. Histogram of measured passage times $P(T)$ of the $9/3$ state showing a comparison between the experimental results and the analytical distribution $P(T)$. Inset: A comparison between the tail of the distribution and an exponential fit for a bin size of 20 frames.

is superposed on the experimental distribution of Fig. 4. $P(T)$ has three parameters: λ, ϵ, and δ. The graph uses the experimentally determined value of λ and was found to be relatively insensitive to the selection of the other two parameters.

The small number of cells plays an important role in this observation of intermittently ordered states. A small number of cells forces a wider separation in parameter space between the stable states of the system, and therefore provides a significant range over which the intermittent ordered states are observed. The symmetry of the experiment creates a situation in which the heteroclinic orbits persist over a finite range of parameters, rather than a measure zero set. This feature was noticed by Aubry *et al.* for $O(2)$ symmetry in a model for boundary layer dynamics [16], by Armbruster, Guckenheimer, and Holmes for the one-dimensional Kuramoto-Sivashinsky equation [17], and by Proctor and Jones in truncated models of Rayleigh-Bénard convection [18].

In this paper we have presented a discussion of the characteristics of pattern-forming systems exhibiting heteroclinic connections, emphasizing the importance of the passage time of the intermittent pattern and its relationship to the physics of unstable equilibria. A skewed distribution of passage times with exponential tail is a distinctive signature of systems in which a trajectory is repeatedly injected near the stable manifold of a saddle node in the presence of noise.

A specific criterion for measuring the passage times in a pattern-forming system was developed, and a quantitative comparison was made between the theoretical expression for the form of the distribution of these times and experimental measurements. The results presented here are representative of seven other points we have investigated in propane-air cellular flames. They provide substantial evidence that the intermittently ordered states arise because of the presence of a network of heteroclinic connections among unstable equilibria.

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