# Suppression of Chaos in a Simplified Nonlinear Dynamo Model

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A simplified nonlinear dynamo model is constructed that allows the transition from the kinematic to the dynamic regime to be studied in detail. We apply this construction to a chaotic flow recently studied in the context of fast dynamo action. It is found that the structure of the magnetic field in the two regimes is markedly different. Furthermore, the saturation of the exponential growth of the magnetic field is achieved by a drastic suppression of the chaotic properties of the flow.

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The fast dynamo problem, concerned with the generation of magnetic fields in highly conducting fluids, has of late received considerable attention [1]. Its simplest formulation relies on the kinematic assumption, in which the velocity is prescribed and the magnetic field evolves due to the induction equation. In this framework one attempts to identify those properties of the velocity that lead to field amplification in the limit of vanishing magnetic diffusivity (infinite magnetic Reynolds number).

The kinematic approach is valid when the magnetic field is weak, and indeed is a natural starting point for the problem. However, in reality, the exponential growth of the magnetic field cannot continue indefinitely; eventually the backreaction of the magnetic field (via the Lorentz force) will modify the flow, thus causing the growth to saturate. Exactly how this modification occurs is at the heart of the nonlinear dynamo problem, though at present it is not clearly understood. Furthermore, the chaotic properties of the resulting velocity fields in this regime, where the magnetic field (on average) neither grows nor decays, have only recently been studied [2].

It is recognized that the essence of the kinematic problem is the competition between line stretching leading to field amplification and enhanced diffusion leading to the destruction of magnetic flux [3]. It is natural to regard both effects from the Lagrangian standpoint of considering particle trajectories. Indeed, it is known that a necessary (though not sufficient) condition for fast dynamo action is that the flows possess regions of chaotic trajectories, in which the process of line stretching is exponential [4]. However, for such flows, gradients also increase exponentially and therefore the success or failure of the dynamo depends on the relative importance of these two processes. It is thus important to understand the nature of these two processes in the saturated regime. It is clear that they must balance; this, however, can be achieved in a number of different ways. At one extreme, the stretching of the field remains vigorous and the dissipation is enhanced; at the other, both the stretching and the dissipation become small. These two possibilities describe very different physical situations; in the former, the flow remains chaotic and the dynamo process is strongly dissipative; in the latter, the flow is only weakly chaotic and weakly dissipative. It is likely that the transport properties of these two types of flow are very different.

There are various approaches to gain insight into the behavior of nonlinear dynamos. One possibility is to solve the equations of nonlinear magnetohydrodynamics (MHD) [5]. In this case the velocity is no longer prescribed, but occurs as a solution of the Navier-Stokes equation including the Lorentz force. The dynamical state in which the dynamo saturates then emerges naturally as a self-consistent solution of the full equations. Clearly this approach contains all the relevant physics. However, due to the inherently three-dimensional nature of the dynamo problem, a fully nonlinear treatment becomes extremely computationally expensive, even for moderate magnetic Reynolds numbers. Furthermore, in a fully nonlinear formulation, one is allowed only the *a priori* specification of the forcing function; it is therefore difficult to make contact with the results of kinematic theory, which is based on a description of the velocity. One of the problems is that in most cases the velocities considered by kinematic models become unstable for modest values of the Reynolds number even in the absence of magnetic effects. In other words, it is often hydrodynamically difficult to "force" the fluid to flow in the kinematically required manner.

An alternative approach, which we shall pursue here, is to concentrate on the velocity field, at the expense of sacrificing some of the physics. One of the motivating factors for our approach is that the (purely hydrodynamical) instabilities alluded to above are related to the inertial term in the momentum equation. If the latter is neglected, the momentum equation becomes linear in velocity which can then be divided into two ingredients, one that is prescribed (as in kinematic theory) and the other driven solely from the action of the Lorentz force. The first can be chosen to lead to the exponential growth of the magnetic field while the second brings about its saturation. The combined velocity field has the interesting property of describing a stationary dynamo.

The governing equations of MHD in dimensionless units are [6]

$$(\partial_t - R_m^{-1}\nabla^2)\mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}), \tag{1}$$

$$(\partial_t - R_e^{-1}\nabla^2)\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} = -\boldsymbol{\nabla}p + \mathbf{J} \times \mathbf{B} + \mathbf{F}, \tag{2}$$

where $\mathbf{u}$, $\mathbf{B}$, and $p$ are the velocity, magnetic field, and pressure, respectively; $\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B}$ is the electric current and $\mathbf{F}$ a forcing function. The dimensionless quantities $R_e$ and $R_m$ are the kinetic and magnetic Reynolds numbers. By definition $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and we assume that $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$.

If we neglect the inertia term in the momentum equation and write

$$\mathbf{u} = \mathbf{u}_o + \mathbf{u}_i, \tag{3}$$

then (2) can be separated into

$$(\partial_t - R_e^{-1}\nabla^2)\mathbf{u}_o = -\boldsymbol{\nabla}p_o + \mathbf{F}, \tag{4}$$

$$(\partial_t - R_e^{-1}\nabla^2)\mathbf{u}_i = -\boldsymbol{\nabla}p_i + \mathbf{J} \times \mathbf{B}. \tag{5}$$

Formally, the velocity $\mathbf{u}_o$ is defined in terms of the forcing function $\mathbf{F}$; however, for our purposes it is more instructive to regard it as prescribed. The induced velocity $\mathbf{u}_i$ is determined by the backreaction due to the Lorentz force. The idea is to choose $\mathbf{u}_o$ to have known interesting properties from kinematic dynamo theory and to evolve (1), (4), and (5) to get a modified velocity $\mathbf{u}_o + \mathbf{u}_i$ that incorporates aspects of both the kinematic and dynamical regimes. The decomposition $\mathbf{u}$ into a driving and an induced component can be rigorously justified only if $R_e \ll 1$. Nonetheless, even when this inequality is not strictly valid, we believe that the construction above will provide useful insights into the nonlinear dynamo problem. Indeed, a similar approach has been used successfully in studies of nonlinear magnetoconvection [7]. Clearly, however, the present model will be inappropiate in cases where the saturation mechanism depends crucially on a (forward or backward) cascade of kinetic energy.

We now apply the procedure above to the flow

$$\mathbf{u}_o = (\partial_y\psi, -\partial_x\psi, \psi), \tag{6}$$

$$\psi = \sqrt{3/2}\,[\sin(x + \cos t) + \cos(y + \sin t)]. \tag{7}$$

This flow has large regions of chaotic streamlines and its kinematic fast dynamo properties have been extensively studied [8,9]. Since this velocity is $z$ independent, we can seek monochromatic solutions of the form

$$\mathbf{B}(\mathbf{x}, t) = \hat{\mathbf{B}}(x, y, t)\exp(ikz). \tag{8}$$

With such a magnetic field, the velocity induced by the Lorentz force will not be $z$ independent; indeed the problem quickly becomes fully three dimensional. Thus, in order to retain the simplifying assumptions of the kinematic theory into the dynamical regime we make the further assumption that $\mathbf{u}_i$ is also independent of $z$. In practice, this can be achieved by retaining only the $z$-averaged component of the Lorentz force. Under this assumption the velocity $\mathbf{u}_i$ is then determined by

$$(\partial_t - R_e^{-1}\nabla^2)\mathbf{u}_i = -\boldsymbol{\nabla}p_i + \langle\mathbf{J} \times \mathbf{B}\rangle_z, \qquad \boldsymbol{\nabla} \cdot \mathbf{u}_i = 0. \tag{9}$$

This is a good time to remark on the nature of the driving and induced velocities $\mathbf{u}_o$ and $\mathbf{u}_i$. One possibility that would lead to saturation is that $\mathbf{u}_o \approx -\mathbf{u}_i$, i.e., the effect of the Lorentz force is simply to reduce the amplitude of the overall flow. However, for large $R_m$ this is unlikely to be the case. Assuming that $\mathbf{u}_o$ is a large scale flow of characteristic size $\ell$ then the resulting magnetic field will have fluctuations down to (small) scales of order $R_m^{-1/2}\ell$; consequently, the resulting induced flow $\mathbf{u}_i$ will likewise have small scale components. Recent work has shown that often it is this small scale component that is responsible for the dynamical saturation of the dynamo process [10]. For this reason, although the truncation in $z$ is severe (a single mode), we solve Eq. (9) with high resolution in the $x$-$y$ plane.

We have followed the evolution of this system numerically, starting with a weak magnetic field; for the results displayed here $R_e = R_m/4$ and $k = 0.57$ [8,11]. Initially the system is in the kinematic phase; $\mathbf{u}_i$ is negligible compared to $\mathbf{u}_o$ and, after a few turnover times, the magnetic field develops a well-defined eigenfunction with an amplitude that grows exponentially [8]. The kinematic phase ends when the peak magnetic field becomes comparable with the velocity. By now the induced velocity $\mathbf{u}_i$ is, at least somewhere, of the same magnitude as the driving flow $\mathbf{u}_o$. There follows a period of readjustment characterized by an increase in the scale of magnetic dissipation defined by $\sqrt{\langle\mathbf{B}^2\rangle/\langle\mathbf{J}^2\rangle}$. Eventually the system settles down to a time-dependent stationary state, in which, on average, the field neither grows nor decays.

The exponential growth and eventual saturation can clearly be identified in Fig. 1, which shows the temporal evolution of the magnetic energy $\langle\mathbf{B}^2/2\rangle$. The stationary state value is comparable to the kinetic energy density (equipartition). The transition phase from the kinematic to the dynamical regime is best seen in Fig. 2, which shows the dissipation scale for the magnetic field for three values of $R_m$. In the kinematic phase (up to $t \approx 30$ in all three cases) the dissipation scale is roughly constant. The initial rapid decrease, barely visible on this scale, results from the initial conditions. The saturation phase starts with an abrupt increase in scale, followed by a further period of readjustment, before reaching the final stationary state. In both the kinematic and dynamical regimes the dissipation scale decreases with increasing $R_m$.

It is natural to enquire into the relation between the magnetic field in the kinematic and final stationary states. One possibility is that by and large the structure of the field remains the same, except possibly for an increase in the

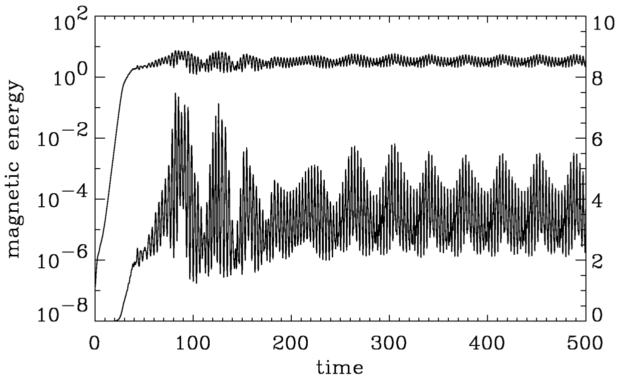


FIG. 1. Temporal evolution of the magnetic energy; $R_m = 100$. The upper curve is logarithmic (corresponding to the left-hand axis), the lower curve is linear (corresponding to the right-hand axis). The kinematic phase, during which the field grows exponentially and the backreaction is negligible, ends at $t \approx 30$. The saturation phase extends from $t \approx 30$ to $t \approx 250$, after which the dynamo is in its final stationary state.

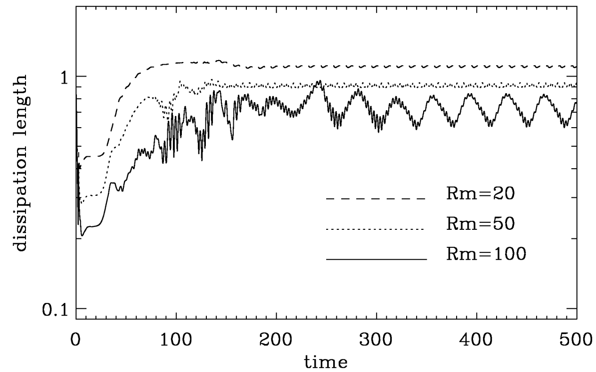


FIG. 2. Temporal evolution of the magnetic dissipation length scale for three different values of $R_m$.

dissipation scale (see Fig. 2). In this case the saturation process merely limits the amplitude of the magnetic eigenfunction, but with no significant restructuring of the magnetic field. The other possibility is that in the saturation process nonlinear interactions lead to a substantial change in the velocity with a corresponding restructuring of the magnetic field. In this case the final magnetic field bears little resemblance to the field in the kinematic regime. The evidence from our computation shows that the second possibility is in fact the correct one. This point is illustrated in Fig. 3 which shows the current distribution in the kinematic and dynamical regimes at three corresponding times during a cycle of the velocity $\mathbf{u}_o$. It is clear that the structure of the magnetic field as well as the relation between the field and the driving velocity are dramatically different in the two regimes.

The saturation process must necessarily also involve a modification of the velocity so that in the final stationary state some balance is reached between the rates of field generation and dissipation. The changes, however, are subtle and not such that they can be illustrated by merely looking at the (Eulerian) structure of the velocity field (for example, its rms velocity remains roughly the same). The nature of the changes becomes apparent only by examining

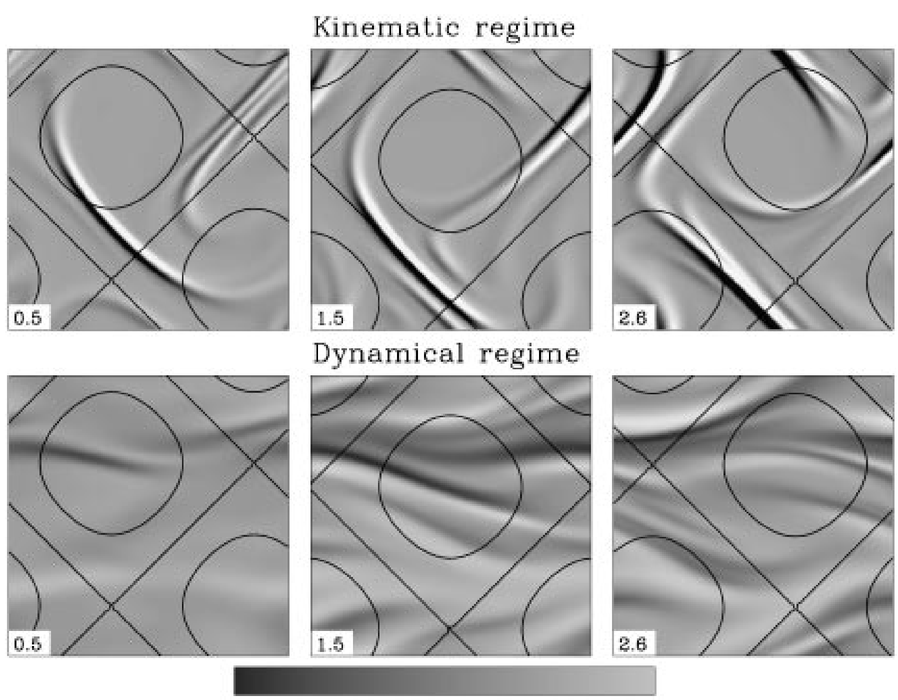


FIG. 3. Plots of $\mathbf{J} \cdot \hat{\mathbf{z}}$ for the kinematic and dynamical regimes on the plane $z = 0$ ($R_m = 100$). The dark lines are contours of $\psi$, corresponding to projections of the streamlines of $\mathbf{u}_0$ onto any $x$-$y$ plane. They are included to show the relation between the magnetic structure and the driving velocity at three corresponding phases of $\mathbf{u}_0$. The upper and lower sets have been scaled independently.

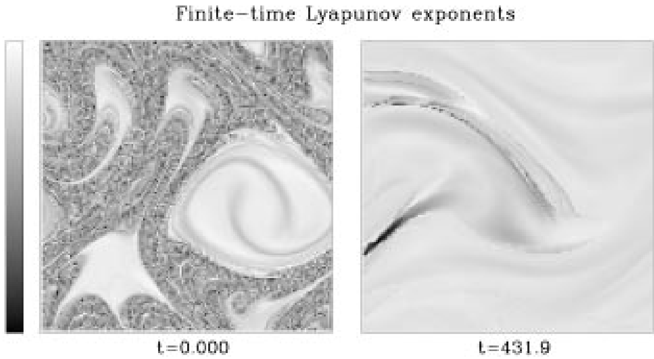


FIG. 4. Spatial distribution of finite-time Lyapunov exponents, starting from the indicated times. The shades code the values of the exponents as a function of the initial positions. Light tones correspond to trajectories with little or no (exponential) stretching; dark tones correspond to strongly stretching trajectories. Regions of chaotic motion, which occupy a substantial fraction of the domain in the kinematic regime, are almost completely absent in the later dynamical phases.

the Lagrangian properties of the flow. We have thus computed the finite-time Lyapunov exponents [12] for the total velocity $\mathbf{u} = \mathbf{u}_o + \mathbf{u}_i$ in the kinematic and dynamical regimes (Fig. 4). This was achieved by following for 25 time units $128^2$ trajectories initially distributed uniformly over the $z = 0$ plane and determining the average (exponential) stretching rate along each trajectory. In the kinematic regime there are large regions of chaotic trajectories along which fluid elements and therefore magnetic field lines are stretched exponentially; in the dynamical regime chaotic regions are almost completely absent.

The foregoing analysis provides a valuable insight into the nature of the dynamo saturation process. Equilibration is brought about by a suppression of the chaotic stretching of the field *and* a corresponding reduction in the rate of magnetic field dissipation. This should be contrasted with the alternative scenario in which stretching remains vigorous but there is a corresponding increase in the efficiency of the dissipation [2]. Finally, these results illustrate the intrinsic limitations of kinematic theory. In order to understand magnetic fields in realistic physical situations it is necessary not only to understand their growth but also the processes by which they stop growing, since it is these that eventually determine the structure of the observed magnetic fields.

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