Negative Resistance and Rectification in Brownian Transport

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We discuss under what conditions Brownian transport processes can display negative resistance. We prove it cannot occur on 1D spaces like the circle or the line. We construct an entropic ratchet: an explicit two-dimensional model, and its collapse onto a branched 1D backbone, showing negative resistance and rectification as a consequence of a geometric symmetry breaking. We derive an accurate numerical method for solving our 2D model. Finally, we discuss analogies and relevance to biological ion channels, in particular, for channel inactivation and blocking.

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If a current arises as a result of some force, then it will flow "downhill," in the direction in which it dissipates energy into heat; thus, the resistance is always positive. The diminishing of a current as the driving force becomes stronger is called negative incremental resistance, or just plain "negative resistance" (NR); devices that display NR exist and have important technological applications; these devices are, usually, also rectifiers. If energy is provided through suitable bias voltages, so as to displace the steady state of operation to the NR region, then the negative slope of the current can give rise to interesting instabilities like relaxation oscillations [1,2]. A typical such device from electronics is the tunnel diode. Much more importantly, the very existence of our nervous systems depends upon the ability to generate action potentials to transmit nerve impulses along axons; this phenomenon requires at least one NR rectifying device, which is known to be the sodium channel [3–7].

In this Letter, we will show how NR and rectification can arise in a Brownian transport process, through a mechanism we will call *entropic trapping.* Because of the purely geometric nature of this mechanism, we can call our model an *entropic ratchet.*

We will study a Brownian particle in a periodic potential, embedded in an equilibrium, constant temperature bath, and subject to a single force trying to advect it. This type of process is described by a Langevin equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \xi(t),$$

where $\mathbf{x}$ belongs to some Euclidean space, ξ is Gaussian white noise satisfying $\langle \xi_i(t)\xi_j(s)\rangle = 2kT\delta_{ij}\delta(t-s)$, and $\mathbf{f}$ is of the form $-\nabla V(\mathbf{x}) + \mathbf{F}$ with $\mathbf{F}$ constant; i.e., $\mathbf{f}$ is a vector field independent of time. We will assume V to be periodic along the direction of $\mathbf{F}$.

Why would it be interesting to search for NR in transport processes of this form? First of all, these processes have very wide applications [8]. Second, transport in symmetry broken potentials ("ratchets") [8–14] has been shown to be analogous to conduction through electronic diodes, capable of rectification; however, the mechanism through which a *tunnel* diode provides NR is intrinsically quantum mechanical; hence, it would be interesting to provide classical analog. Finally, ratchet potentials in one dimension can be shown not to have NR, as we will now do.

In one dimension, an equation of the form

$$\dot{x} = f(x) + \xi(t), \qquad \langle \xi(t)\xi(s)\rangle = 2kT\delta(t-s)$$

has an associated stationary Fokker-Planck equation

$$\partial_t P(x,t) + \partial_x J(x,t) = 0,$$
$$J(x,t) = fP(x,t) - kt\partial_x P(x,t),$$

whose steady state, for periodic $f(x)$, can always be solved in quadratures. If f has a zero spatial average, then it is the gradient of a periodic potential, and detailed balance and Boltzmann weights hold. If f does not have a zero average, then it can be written as $f = -\partial_x V + F$ with V periodic, and the stationary state can be solved in double quadratures [11]. The Fokker-Planck probability current J as a function of F is given by

$$J(F) = \frac{kT(e_{kT}^{2\pi F} - 1)}{Q(F)}, \tag{1}$$

$$Q(F) = \oint\oint e_{kT}^{V(x')-V(x)+F(x-x')+2\pi F\Theta(x'-x)}\,dx'\,dx, \tag{2}$$

where e_{kT}^{x} means $\exp(x/kT)$. Q is positive; moreover, $\partial_F Q$ is also positive, and satisfies $kT\partial_F Q < 2\pi Q$. This inequality implies $dJ/dF > 0$ and hence Brownian transport in one-dimensional periodic potentials (under a steady force and in white noise) cannot show NR. (We will show later that this only holds if the underlying space is topologically trivial.)

Now we will construct an explicit example in two dimensions. First we illustrate the notion of entropic barriers, through a very simple example. Let us consider the following potential:

$$V(x,y) = x^2[\cos(y) + 1.1].$$

We see that the potential is identically zero on the y axis, and bigger than zero everywhere else. Hence there are no true energy barriers impeding motion of a Brownian particle along the periodic direction y. However, the shape

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of the potential around the y axis is also important at nonzero temperature. For any given y, a slice of the potential along that value of y is a parabola; however, as motion progresses along y this parabola opens and closes periodically. In the absence of an external force, a Brownian particle will spend more time around $y = \pi$ than on the bottleneck $y = 0$, every now and then jumping one period up or down in y, as if there actually were an energy barrier; these are called *entropic barriers* because, unlike a true activation energy, the time scales they induce do not follow Arrhenius-Kramers laws. In other words, as we will see, the violation of these laws is responsible for the nonmonotonic behavior of the current as a function of the applied force F.

We will now break the parity symmetry along the y axis; if we loosely call any parity broken potential a "ratchet," we can say the following potential is an *entropic ratchet:*

$$V(x, y) = x^2[\cos(y + \ln\cosh x) + 1.1]/2\,, \qquad (3)$$

where $\ln\cosh x$ is just an easy way to make a function that is both even and linear in x for large x. The equipotentials are now symmetry broken and look like a herringbone pattern; see Fig. 1. Our dynamics will be given by

$$\dot{\mathbf{x}} = -\nabla V + F\hat{\mathbf{y}} + \xi(t)\,, \qquad (4)$$

where $\hat{\mathbf{y}}$ is the unit vector along y, and the noise correlators are as before.

So, if we apply a force F along y, if the force is positive the particle will move forward without problem. However, if the force is negative, the particle will move backwards, but every now and then it will get into a spine of the herringbone. It will progress upon the spine for a while, deeper the stronger the force, and then will have to go *against* the force in order to climb back up and get again on the backbone. Thus it will be locked for a while, because the energy required to climb back up is an honest activation energy, and the time required to do so obeys an Arrhenius law. But the probability that a particle on the backbone will get into the spine does not depend exponentially on F, and so the particle spends a larger proportion of time blocked away in the spines as F becomes more negative. The overall time scale to get into and out of the spine is not Arrhenius-like, yet near the bottleneck the potential energy of points in the spine and in the backbone is the same. If it was, the time the particle spends stuck in the spine would cancel out with the time it spends in the backbone, and the current would be a monotonic function of F.

We have evolved numerically Eq. (4) to obtain the average speed of the particle as a function of F and kT. The mean speed equals the Fokker-Planck current times the period (2π) of the potential. The result of our simulations is shown in Fig. 2.

In performing a numerical integration of Eq. (4), one encounters several numerical problems. The simplest method for numerical integration is the Euler method,

$$x(t + \Delta t) = x(t) + f(x)\Delta t + \sqrt{2kT\Delta t}\,\eta\,,$$

where the η are random Gaussian numbers with unit variance. While the Euler method is first order in Δt for an ordinary differential equation, it is only one-half order for a Langevin equation, a property characteristic of diffusion. This implies that extremely small time steps have to be used for accurate integration. Several methods have been proposed for increasing the order of the integration, including stochastic versions of Runge and Kutta [15]. However, the spines in our potential are a *stiff* problem, being long and skinny, which would limit the applicability of a Runge-Kutta scheme even in the absence of noise. The reason is that the time scale has to be small in comparison with the relaxation time on the fastest direction, in this

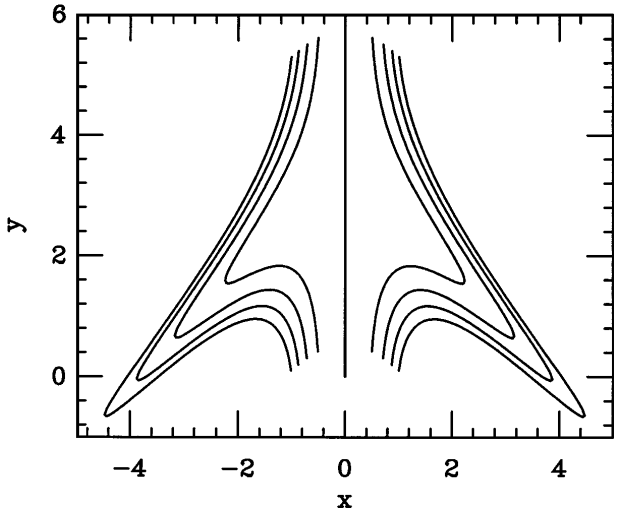

FIG. 1. The equipotentials of the model.

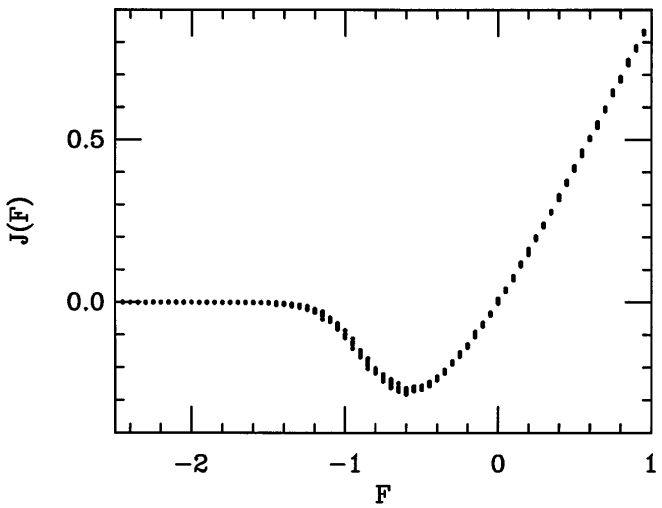

FIG. 2. $J(F)$ at $kT = 1$ as computed from Langevin simulations. Each run lasted for 10^5 units of time; 10 runs were done for each value of F.

case the skinny direction normal to the axis of the spine. But then the interesting time evolution is that *along* the axis of the spines, which becomes painfully slow.

We have devised a method to cope with this problem. The fundamental problem in developing a Runge-Kutta scheme is that somehow one is assuming analyticity both of the vector field and of the solution; the last one is just not there. But one can safely assume analyticity of the vector field alone. If we expand the latter to first order,

$$\mathbf{f}(\mathbf{x} + \mathbf{\Delta x}) \approx \mathbf{f}(\mathbf{x}) + \mathbf{\Delta x} \cdot (\mathbf{\nabla f})(\mathbf{x}) .$$

We use the fact that $\mathbf{f}$ is curl free (and thus $\mathbf{\nabla f}$ is a symmetric tensor) to diagonalize it and rotate to its eigenbasis. In this base, the problem locally becomes a cross product of independent Ornstein-Uhlenbeck processes. Thus, the question is: if we know we currently are at position x at time t, what is the probability that we will be at position x' at time t'? The answer to this question is the Fokker-Planck propagator (Green function), which is known analytically for the Ornstein-Uhlenbeck process [8]. It is a Gaussian, centered at the position x', that would be the solution for the deterministic case at time t', and with a width which has been changed because of the compression or expansion due to $f' \equiv df/dx$. Thus it is a trivial matter to generate a new value x' with the correct probability distribution for an Ornstein-Uhlenbeck process; our numerical method then reads

$$x' = (x - x_c)e^{-f'\Delta t} + x_c + [(kT/f')(1 - e^{-2f'\Delta t})]^{1/2}\eta ,$$
$$x_c = x - f/f', \qquad \Delta t = t' - t ,$$

for each eigendirection of the Hessian. The new coordinates are then rotated back to the original frame. The advantages of this method over the stochastic Runge-Kutta method are two. First, it solves *exactly* the Ornstein-Uhlenbeck process, by construction, and hence any linear problem. Second, it can handle stiff problems more easily; it will not lose accuracy if the time step is larger than the relaxation time scale of the fast direction, because it will not *overshoot* and generate dynamical instabilities; one can concentrate on the more interesting slow time scale. The disadvantages are also two: analytic knowledge of the Hessian is required, and a matrix eigensystem calculation has to be performed, together with two coordinate frame transformations. Thus our method rapidly loses ground to the stochastic Runge-Kutta method for high dimensionalities, unless the problem is quite stiff or the Hessian is sparse. For our stiff problem in two dimensions, this method is extremely well adapted; Fig. 2 was generated in a few hours of CPU time of a workstation and shows almost no trace of the residual noise typical of direct Langevin simulations.

Back to our problem, we would like to get some more explicit understanding of the $J(F)$ curve we have just obtained numerically. Let us consider a one-dimensional cartoon of the system: we can represent the backbone as a circle, with a single spine of length L attached to it, at an angle θ, as in Fig. 3(a). There is no potential, just the steady force. At the vertex, we have to impose that the probability be continuous: the limit of $P(x)$ as we approach the vertex from all *three* sides must be the same, P_{vert}. The current should be conserved at the vertex, so that probability does not accumulate. Then, the probability density on the circle is constant, and so is the current on the backbone: $J = FP_{\text{vert}}$. The current on the spine should vanish, and hence the probability along the spine $P(x) \approx \exp[-Fx/\tan(\theta)]$. Therefore, the final current is

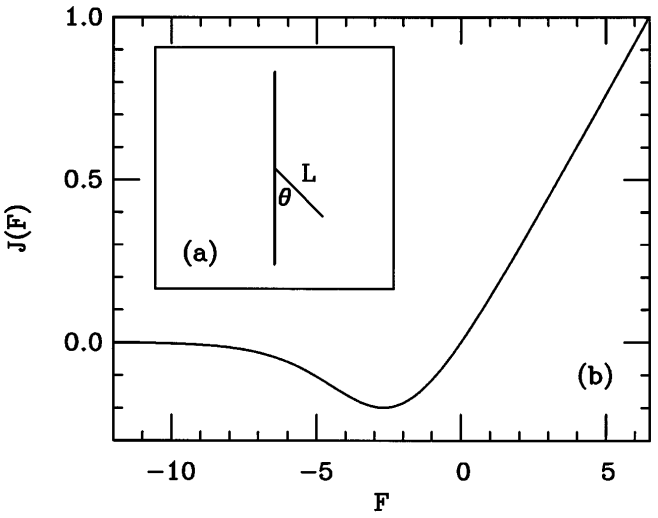

FIG. 3. (a) The phase space for the cartoon. (b) $J(F)$ as given by Eq. (4).

$$\frac{F}{J(F)} = 2\pi + \frac{kT}{F\cos\theta}(1 - e_{kT}^{-FL\cos\theta}) , \qquad (5)$$

which has the right qualitative form [Fig. 3(b)], except for a single detail. For F large and positive both cartoon and full case converge algebraically to unity. But for F large and negative, $J \to 0$ exponentially, while in the full case it seems to die faster. This behavior can be understood if we recall that the spines are truly different: first, there is a potential along them; second, they are not of length L, but rather arbitrarily deep. The center of the spines lies approximately at $y = \pi - \ln\cosh x$, and the potential there is $V = 0.05x^2$. The current in this case equals

$$\frac{F}{J(F)} = 2\pi + 2\int_0^\infty \sqrt{1 + \tanh^2(x)}\, e^{-0.05x^2 + F\ln\cos(x)}\, dx . \qquad (6)$$

Thus, for a large and negative force, the particle entering the spine will find a stable fixed point at $x \approx 10F$. Thus the effective length of the spine *increases* as F becomes more negative; the activation barrier is the product of this effective length and F, and hence *quadratic* in F rather than linear. So we have shown that NR is actually possible in one dimension, but only if the topology of the space is more complicated than just a circle.

We have presented an explicit example of a Brownian transport process showing NR; the NR region is generated through an essentially entropic process. There are no obstacles and no energy barriers to the motion of the particle along the y axis. There is just a finite probability of exploring space a bit out of the center, and end up blocked inside a spine. Since we are in a thermal, rather than quantum, situation, our system has much more similarity to biological ion channels than to electronics. There are two well known instances in channels where NR is observed. The first one is in channels that can have an *inactive* state; this is the case of the Na channel, but some K channels also have this property; this is extremely important biologically, because the NR resulting from inactivation is essential to the regeneration of action potentials. The second one is the blocking of channels through large ions; this is important experimentally, because in order to assess properties of new channels, biophysicists will test for changes in behavior when the channel is "poisoned" with various compounds of known effect on known channels. The standard poison arsenal includes several large ions, that can get partially into the channel and block it; for example, tetraethylammonium is used for Na channels [16] and Mg [17], Cs [18], or polyamines [19] for K channels. It is worth noting that the *I-V* curves of such channels look extremely similar to that of our model, and, furthermore, several of the experimental measurements [6,16,18] show the faster-than-exponential decay of our model, while standard theoretical models with *fixed* barriers [17,20] show exponential decays; this discrepancy can only be solved through models having barriers that depend on the field. This might mean that the large ion buries itself deeper and deeper into the crevice of the channel, getting more and more stuck and having to climb a larger distance against the potential to get out, just as for our spines. There are also similarities with the case of channel inactivation. These channels are hypothesized to have three states: closed, open, and inactive. The transition between open and inactive states has been modeled with a "ball and thread" mechanism [7,20,21]: some mobile, charged, globular piece of the channel loosely attached through a long thread can get stuck in the mouth of the channel and block it; pretty much like a large ion blocker. Our model does not include a closed state, since the central backbone is always "open"; thus we do not see the current going to zero as F becomes large and positive. The closed state can be bypassed by looking at the peak sodium current from a pulse rather than the stationary current [22], and in this case the observed peak *I-V* curve is qualitatively similar to what we observe.

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