Generalized Synchronization, Predictability, and Equivalence of Unidirectionally Coupled Dynamical Systems

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Necessary and sufficient conditions for the occurrence of generalized synchronization of unidirectionally coupled dynamical systems are given in terms of asymptotic stability. The relation between generalized synchronization, predictability, and equivalence of dynamical systems is discussed. All theoretical results are illustrated by analytical and numerical examples. In particular, the existence of generalized synchronization in the case of parameter mismatch between coupled systems leads to a new interpretation of recent experimental results. Furthermore, the possible application of generalized synchronization for attractor reconstruction in nonlinear time series analysis is discussed.

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Since 1990 chaos synchronization has been a topic of great attention (see [1–10] and references cited therein). Usually two dynamical systems are called *synchronized* if the distance between their states converges to zero for $t \to \infty$. Recently [8], a generalization of this concept for unidirectionally coupled systems was proposed, where two systems are called synchronized if a (static) functional relation exists between the states of both systems. In [8] this kind of synchronization was called *generalized synchronization* (GS) and a numerical method (called mutual false nearest neighbors) was proposed for detecting the presence of the functional relation between the states of the coupled systems.

The main goal of this Letter is to develop a general theory for GS of unidirectionally coupled systems. In particular, we give conditions for the occurrence of GS and discuss its relation to predictability and equivalence of chaotic systems. The statements are illustrated using analytical and numerical examples. Furthermore, we show that many real experiments of chaos synchronization, in which nonsynchronization has been observed due to mismatch of the parameters are actually examples for GS.

An important class of synchronizing systems is unidirectionally coupled systems (master-slave configurations, or systems with a skew product structure):

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{y}, \mathbf{u}) = \mathbf{g}(\mathbf{y}, \mathbf{h}(\mathbf{x})), \end{aligned} \tag{1}$$

where $\mathbf{x} \in R^n$, $\mathbf{y} \in R^m$, and $\mathbf{u}(t) = (u_1(t), \dots, u_k(t))$ with $u_j(t) = h_j(\mathbf{x}(t, \mathbf{x}_0))$. The first and second systems in (1) are referred to as a *drive* and *response*, respectively. Here the variables u_j are introduced to include explicitly the case that a function $\mathbf{u} = \mathbf{h}(\mathbf{x})$ of $\mathbf{x}$ is used for driving the response system. We say that (1) possesses the property of GS [2,8] between $\mathbf{x}$ and $\mathbf{y}$ if there exists a transformation $\mathbf{H}: R^n \to R^m$, a manifold $M = \{(\mathbf{x}, \mathbf{y}) : \mathbf{y} = \mathbf{H}(\mathbf{x})\}$, and a subset $B = B_x \times B_y \subset R^n \times R^m$ with $M \subset B$ such that all trajectories of (1) with initial conditions in the basin B approach M as time goes to infinity. If $\mathbf{H}$ equals the identity transformation, this general definition of synchronization coincides with the usual definition given in the introduction. This case will be referred to as *identical synchronization* (IS) in the following.

Directional coupling was intensely studied in combination with different methods for constructing synchronized systems [1–10]. It may be viewed as a generalization of periodic or quasiperiodic driving that has been used in physics, mathematics, and engineering for a long time. Furthermore, unidirectionally coupled systems may lead to interesting applications, for example, in communication systems [5,9,10]. Rulkov *et al.* [8] have presented examples for GS in unidirectionally coupled systems, where the transformation $\mathbf{H}$ is a known vector valued function. Rulkov [11] also suggests a simple way for detection of GS by plotting a variable of the response system versus the same variable of a second, identical response system starting from different initial conditions. In the case of GS the resulting curve converges to the diagonal.

In this Letter we address the question: "Under what conditions does GS occur for the unidirectionally coupled system (1)?" The main result giving an answer to this question is the following theorem.

Theorem: GS occurs in system (1) if and only if for all $(\mathbf{x}_0, \mathbf{y}_0) \in B$ the driven system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, \mathbf{u}) = \mathbf{g}(\mathbf{y}, \mathbf{h}(\mathbf{x}))$ is asymptotically stable [i.e., $\forall \mathbf{y}_{10}, \mathbf{y}_{20} \in B_y$: $\lim_{t\to\infty} \|\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_{10}) - \mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_{20})\| = 0$].

Proof: Let $\phi_x^t : R^n \to R^n$ be the flow of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\Phi^t = (\phi_x^t, \phi_y^t)$ the flow of (1) with $\phi_y^t : R^{n+m} \to R^m$. In order to construct the map $\mathbf{H}$ explicitly we choose an arbitrary point $\mathbf{x}_0 \in B_x$ and determine the corresponding image point $\mathbf{y}_0 = \mathbf{H}(\mathbf{x}_0)$. Since all states $\mathbf{y} \in B_y$ of the response system converge only asymptotically to the manifold M we consider trajectories starting in the past at the point $(\phi^{-t}(\mathbf{x}_0), \mathbf{y}_0)$. When this trajectory passes the point $\mathbf{x}_0$ the time t has elapsed and the point $(\mathbf{x}_0, \phi^t(\mathbf{y}_0))$ is the closer to M the larger t is. Formally we define $\tilde{\mathbf{H}}(\mathbf{x}_0, \mathbf{y}_0) = \lim_{t\to\infty} \phi_y^t(\phi_x^{-t}(\mathbf{x}_0), \mathbf{y}_0)$.

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Asymptotic stability implies $\lim_{t\to\infty} \|\phi_y^t(\phi_x^{-t}(\mathbf{x}_0), \mathbf{y}_{10}) - \phi_y^t(\phi_x^{-t}(\mathbf{x}_0), \mathbf{y}_{20})\| \to 0$ for all $\mathbf{y}_{10}, \mathbf{y}_{20} \in B_y$, and therefore $\tilde{\mathbf{H}}(\mathbf{x}_0, \mathbf{y}_0)$ is independent of $\mathbf{y}_0$. The transformation $\mathbf{H}$ defining the synchronization manifold M is thus given by $\mathbf{H}(\mathbf{x}_0) = \tilde{\mathbf{H}}(\mathbf{x}_0, \mathbf{y}_0)$ for arbitrary $\mathbf{y}_0 \in B_y$. Furthermore, asymptotic stability implies that M is an attracting manifold.

A basic technique for proving asymptotical stability is Lyapunov's direct method. This approach was applied for the first time in chaos synchronization by He and Vaidya [4] (see also [6]). In those cases where it is not possible to find a Lyapunov function, one can numerically compute the conditional Lyapunov exponents of the response which were introduced by Pecora and Carroll [3]. In this case, GS occurs if and only if all conditional Lyapunov exponents of the response are negative.

An immediate consequence of the above theorem is that the response is predictable, because GS implies $\mathbf{y}$ *predictability,* that is the ability to predict the behavior of $\mathbf{y}$, based on the knowledge of $\mathbf{x}$ and $\mathbf{H}$ only. If $\mathbf{H}$ is invertible $\mathbf{x}$ is also predictable from $\mathbf{y}$.

Another concept related to GS is *equivalence.* Two vector fields $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$ are *equivalent* if there exists a C^k diffeomorphism $\mathbf{G}$, which takes orbits of $\mathbf{f}$ to orbits of $\mathbf{g}$, preserving the senses but not necessarily parametrization by time. Analogously we call a nonautonomous vector field $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, \mathbf{h}(\mathbf{x}))$ *equivalent* to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if there exists a C^k diffeomorphism $\mathbf{G}$, which takes orbits of $\mathbf{f}$ to orbits of $\mathbf{g}$. This equivalence, of course, holds only if $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, \mathbf{h}(\mathbf{x}))$ is driven by a solution $\mathbf{x}(t)$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and is therefore a *conditional* equivalence.

In some cases the transformation $\mathbf{H}$ that is given by the GS is a diffeomorphism, and thus an equivalence relation between the drive and the response is established by the synchronization. In general, however, GS and equivalence are related but independent notions. This fact and the above theorem will now be illustrated by analytical and numerical examples.

The first example shows that GS can occur for pairs of arbitrary systems provided the response is stable. Here a Lorenz system [12] is driven by a Rössler system [13]. The equations of the drive system are

$$\begin{aligned} \dot{x}_1 &= 2 + x_1(x_2 - 4), \\ \dot{x}_2 &= -x_1 - x_3, \\ \dot{x}_3 &= x_2 + 0.45x_3, \end{aligned} \tag{2}$$

and the response system is given by

$$\begin{aligned} \dot{y}_1 &= -\sigma(y_1 - y_2), \\ \dot{y}_2 &= ru(t) - y_2 - u(t)y_3, \\ \dot{y}_3 &= u(t)y_2 - by_3, \end{aligned} \tag{3}$$

where $u(t)$ is an *arbitrary* scalar function of x_1, x_2, x_3, and $\sigma, b > 0$. In order to show that (2) and (3) are GS we consider the difference $\mathbf{e} = \mathbf{y} - \mathbf{y}'$, where the primed variables are from an identical copy of the response system. Using the Lyapunov function

$$L = (e_1^2/\sigma + e_2^2 + e_3^2)/2,$$

one obtains

$$\begin{aligned} \dot{L} &= -e_1^2 + e_1 e_2 - e_2^2 - be_3^2 \\ &= -(e_1 - e_2/2)^2 - 3e_2^2/4 - be_3^2 < 0, \end{aligned}$$

i.e., the response system is asymptotical stable for *arbitrary* drive signals u and *arbitrary* initial conditions. Therefore GS always occurs although drive and response are completely different systems. Figure 1 shows attractors from (2) and (3) for the case $u = x_1 + x_2 + x_3$. Because of the GS the attractor of the y system (3) shown in Fig. 1(b) is a nonlinear image of the attractor of the x system (2) given in Fig. 1(a). The x_2 vs y_2 diagram in Fig. 1(c) shows that both systems are not synchronized in the sense of IS.

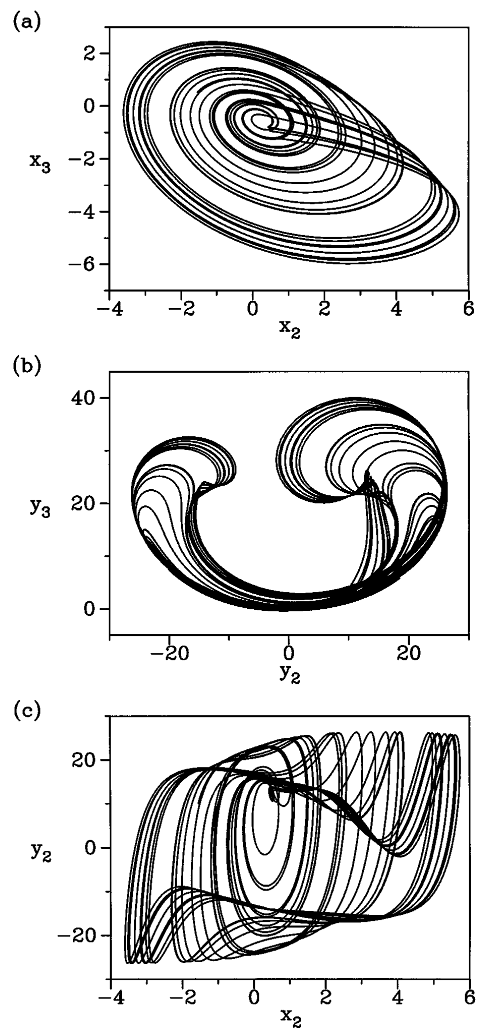

FIG. 1. Generalized synchronization of a Rössler system (drive) and a Lorenz system (response). (a) Rössler x_2 vs x_3. (b) Lorenz y_2 vs x_3 ($\sigma = 10$, $r = 28$, $b = 2.666$). (c) Lorenz x_2 vs Rössler y_2.

With our second example we want to show that equivalence does not imply GS. For this purpose we consider the (x_1, x_2) subsystem [3] of the Lorenz system. The defining equations for the drive are

$$\begin{aligned} \dot{x}_1 &= -\sigma(x_1 - x_2), \\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - bx_3, \end{aligned} \tag{4}$$

and the response system

$$\begin{aligned} \dot{y}_1 &= -\sigma(y_1 - y_2), \\ \dot{y}_2 &= ry_1 - y_2 - y_1u, \\ \dot{y}_3 &= y_1y_2 - by_3 \end{aligned} \tag{5}$$

is driven by $u = x_3$. For a reason that will become clear later on, we have replicated the defining equation for $\dot{y}_3$ in (5). In [3] it is conjectured that for $\sigma = 16$, $b = 4$, and $r = 45.92$ the response system (y_1, y_2) is unstable, since its conditional Lyapunov exponents have been estimated in Ref. [3] to equal $\lambda_1 = +7.89 \times 10^{-3}$ and $\lambda_2 = -17.0$. A more detailed analysis shows, however, that the largest conditional Lyapunov exponent of (5) is zero (this can be proved rigorously [14]) and that the flows of the drive and the response are tightly connected. In other words, $\mathbf{x}(t)$ can be computed directly using $\mathbf{y}(t)$ and the following transformation [15]:

$$y_1 = kx_1, \quad y_2 = kx_2, \quad y_3 = k^2x_3, \tag{6}$$

where k is a constant. In a straightforward manner one can show that the variables of (4) are transformed to those of (5). The constant k depends on the initial conditions $\mathbf{x}_0$ and $\mathbf{y}_0$ of (4) and (5), respectively. At the response, both $x_3(t)$ and $y_3(t)$ are known and k can be computed with sufficient precision after a finite time as $k = \sqrt{y_3/x_3}$. Having calculated k, one can compute x_1 and x_2. Note that the opposite is not possible. Since k (and thus $\mathbf{H}$) is not known at the drive, it is not possible to compute the trajectory of the response only through the knowledge of the trajectory of the drive. Despite the existence of the invertible continuous transformation $\mathbf{H}$, *no* GS occurs between the drive and the response. The reason is that the synchronization manifold $M = \{(\mathbf{x}, \mathbf{y}) : \mathbf{H}(\mathbf{x}) = \mathbf{y}\}$ is not an attractor, because the response system (5) is not asymptotically stable. This example shows that *equivalence does not imply GS.* Namely, (4) and (5) are equivalent [because using transformation (6) x orbits are mapped to y orbits], but they are not GS (because there exists no attracting synchronization manifold).

On the other hand, *GS does not imply equivalence.* This is illustrated by the third example where we choose for the drive system a standard three-dimensional chaos generator (Chua's circuit) [16]:

$$\begin{aligned} \dot{x}_1 &= \alpha x_2 - \alpha[m_1x_1 - m_2(|x_1 + 1| - |x_1 - 1|)], \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -\beta x_2, \end{aligned} \tag{7}$$

where $\alpha = 9$, $\beta = 100/7$, $m_1 = 0.2857$, and $m_2 = 0.2143$. The response is the following one-dimensional system:

$$\dot{y} = -y^3/10 - y(u(t) + p) \tag{8}$$

driven by $u(t) = x_1$. We assume that the drive operates in a chaotic regime, the so-called *double scroll attractor*, for which the average value of x_1 is zero. As will be discussed elsewhere [14], system (7) and (8) can produce on-off intermittency [17] and riddled basins [18]. For $p < 0$ the response system (8) has two attractors located in the regions $y > 0$ and $y < 0$, respectively. It is easy to see that if the initial point $y(0) > 0$, then $y(t) > 0$ for all t. Therefore the corresponding basins of attraction are R^+ and R^-. For $p > 0$, the response possesses only one attractor, the fixed point at the origin. As can be seen in Fig. 2 the conditional Lyapunov exponent of (8) is negative for all values of the parameter p and GS always occurs for the drive and the response. Both systems are not equivalent because $\mathbf{H} : R^3 \to R$ cannot be a diffeomorphism. For large negative p the transformation $\mathbf{H}$ can be approximated by $y = \mathbf{H}(\mathbf{x}) = kx_1 + q$, where $k \to 0$ and $q \to \sqrt{-10p}$ for $p \to -\infty$ [14].

Our fourth example illustrates the influence of parameter mismatch on (generalized) synchronization. Therefore we consider the parameter dependence of the drive and the response explicitly

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}; \mu), \\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{y}, \mathbf{u}; \nu), \end{aligned} \tag{9}$$

where μ and ν denote the corresponding sets of parameters. Suppose that IS occurs for $\nu = \nu_0 = \mu$, and assume that there exists a neighborhood U of ν_0, such that for all $\nu \in U$ the response system is asymptotically stable. In this case, in contrast to IS, the GS is *not* destroyed by the parameter mismatch. As an example consider

$$\begin{aligned} \dot{x}_1 &= -10(x_1 - x_2), \\ \dot{x}_2 &= 28x_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - 2.666x_3 \end{aligned} \tag{10}$$

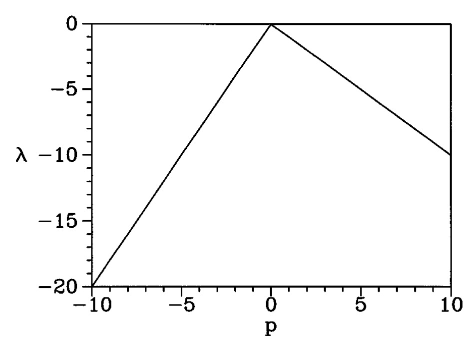

FIG. 2. Conditional Lyapunov exponent of the response system (8) versus parameter p.

and

$$\begin{aligned} \dot{y}_1 &= -\sigma y_1 + u(t)\,, \\ \dot{y}_2 &= r y_1 - y_2 - y_1 y_3\,, \\ \dot{y}_3 &= y_1 y_2 - b y_3\,, \end{aligned} \tag{11}$$

where $u = 10x_2$, and $\sigma > 0, r$ and $b > 0$ are parameters, which are different from the corresponding values in the drive. Define $\mathbf{e} = \mathbf{y} - \mathbf{y}'$, where the primed variables are again from an identical copy of the response system. First we note that e_1 converges to zero, because $\dot{e}_1 = -\sigma e_1$ and $\sigma > 0$. Therefore the remaining two-dimensional system can for the limit $t \to \infty$ be written as

$$\begin{aligned} \dot{e}_2 &= -e_2 - y_1 e_3\,, \\ \dot{e}_3 &= y_1 e_2 - b e_3\,. \end{aligned}$$

Using the Lyapunov function $L = e_2^2 + e_3^2$ one can show that $\dot{L} = -2(e_2^2 + b e_3^2) < 0$. This means that GS occurs for all $\sigma > 0, r$ and $b > 0$, and the attractor of the response is a nonlinear image of the attractor of the drive system (like in the first example, compare Fig. 1). This result shows that in many real experiments [3,7–9] of chaos synchronization GS persists in a certain range of parameters. For example, in the circuit experiment in Ref. [3] the response is a stable linear subsystem. Therefore Fig. 3 in [3] provides a first experimental result of GS. We note that these results also shed new light on the question of robustness of synchronization which, therefore, should be addressed again. In the case of GS, for example, robustness means primarily that the response system remains stable. The question how the loss of stability in the response system affects the synchronization is thus very important [for example, the riddled basins and on-off intermittency of system (7) and (8) are also caused by the instability of the response system [14]]. Moreover, the robustness of GS can also be used, for example, in communication systems. In this case, **H** has to be invertible in order to recover the information signal at the receiver. Using a suitable choice of the function **H** one may even be able to mimic the characteristics of the channel to improve the quality of the received signal [14].

Finally, we discuss the question how GS can be applied to reconstruct attractors from time series. We conjecture that (similar to the case of delay coordinates [19]) it is a generic property of the map **H** to be a diffeomorphism if the dimension of the response system is more than twice the box-counting dimension of the attractor of the drive. Actually, Fig. 1(b) can be interpreted as such a reconstruction of the attractor given in Fig. 1(a). Of course, this kind of embedding would be valid only asymptotically after some synchronization transient has elapsed. Advantages of such a GS reconstruction could be the possibility of real-time hardware implementations and the large variety of reconstructions based on different (passive) response systems.

In conclusion, we have presented a general criteria for the occurrence of generalized synchronization of unidirectionally coupled systems. This criterion is based on asymptotic stability of the response systems, which can be verified using Lyapunov functions or conditional Lyapunov exponents. As an important implication for practical applications it is demonstrated that GS is robust with respect to parameter changes. GS offers also new possibilities for communication schemes using chaotic synchronization. Furthermore, the application of GS for reconstructing attractors from time series has been proposed and discussed. Another interesting question for future research would be whether GS also occurs in synchronization based information processing of neural assemblies [20].

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