Phase Synchronization of Chaotic Oscillators

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We present the new effect of phase synchronization of weakly coupled self-sustained chaotic oscillators. To characterize this phenomenon, we use the analytic signal approach based on the Hilbert transform and partial Poincaré maps. For coupled Rössler attractors, in the synchronous regime the phases are locked, while the amplitudes vary chaotically and are practically uncorrelated. Coupling a chaotic oscillator with a hyperchaotic one, we observe another new type of synchronization, where the frequencies are entrained, while the phase difference is unbounded. A relation between the phase synchronization and the properties of the Lyapunov spectrum is studied.

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Cooperative behavior of chaotic dynamical systems and, in particular, synchronization phenomena have received much attention recently. Nevertheless, the notion of synchronization itself lacks a unique interpretation. Mostly, the synchronization is considered as the complete coincidence of the states of individual systems (subsystems). Such a regime can result from an interaction between systems [1] or subsystems [2,3], as well as from the influence of external noisy [4] or regular [5] fields; in all these situations synchronization is a threshold phenomenon.

Generally, synchronization can be treated as an appearance of some relations between functionals of two processes due to interaction [6]. The choice of the functionals is to some extent arbitrary and depends on the problem under consideration. In the classical case of *periodic* self-sustained oscillators, described as early as in the 17th century by Hugenii [7], synchronization is usually defined as locking of the phases $\phi_{1,2}, n\phi_1 - m\phi_2 =$ const [8], while the amplitudes can be quite different. This effect is widely used in engineering for improvement of the linewidth of a high-power generator with the help of a low-power but more stable (having narrower line) one. Some other types of synchronization in systems with quasiperiodic and chaotic behavior have been discussed in Ref. [9].

In this Letter we investigate phase synchronization of *chaotic* oscillators. Using the methods of analytic signal and the Poincaré map, we show that the interaction of nonidentical autonomous chaotic oscillators can lead to a perfect locking of their phases, whereas their amplitudes remain chaotic and noncorrelated. A similar effect of phase locking of chaotic oscillations by a periodic external force has been described in Refs. [10,11]. We also describe a weaker type of synchronization, when the frequencies are locked while the phase difference exhibits a random-walk-type motion.

Firstly, we have to determine the amplitude and the phase of an arbitrary signal *s(t)*. A general approach has been introduced by Gabor [12] and is based on the analytic signal concept [13]. The analytic signal $\psi(t)$ is a complex function of time defined as

$$\psi(t) = s(t) + j\tilde{s}(t) = A(t)e^{j\phi(t)}, \qquad (1)$$

where the function $\tilde{s}(t)$ is the Hilbert transform of $s(t)$

$$\tilde{s}(t) = \pi^{-1}\text{P.V.}\int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau}\, d\tau \qquad (2)$$

(where P.V. means that the integral is taken in the sense of the Cauchy principal value). The instantaneous amplitude $A(t)$ and the instantaneous phase $\phi(t)$ of the signal $s(t)$ are thus uniquely defined from (1). From (2), the Hilbert transform $\tilde{s}(t)$ of $s(t)$ may be considered as the convolution of the functions $s(t)$ and $1/\pi t$. Hence the Fourier transform $\tilde{S}(j\omega)$ of $\tilde{s}(t)$ is the product of the Fourier transforms of $s(t)$ and $1/\pi t$. For physically relevant frequencies $\omega > 0$, $\tilde{S}(j\omega) = -jS(j\omega)$; i.e., ideally $\tilde{s}(t)$ may be obtained from $s(t)$ by a filter whose amplitude response is unity, and whose phase response is a constant $\pi/2$ lag at all frequencies [13,14].

For chaotic oscillators, we can calculate the phase from taking as $s(t)$ any observable, so there is no unique phase of chaotic oscillations. However, in some cases "natural" observables provide phases that agree with an intuitive definition. For example, for the Rössler attractor [15] taking the observables $s_1 = x$ and $s_2 = y$ [see below Eqs. (3)] gives phases shifted by $\approx\pi/2$ and rotating with the same averaged velocity, corresponding to the main peak in the power spectrum.

To study phase synchronization of coupled chaotic oscillators, we calculate the phases of the oscillators and then check whether the weak locking condition $|n\phi_1 - m\phi_2| <$ const is satisfied. In this Letter, we restrict ourselves to the case $m = n = 1$.

As the simplest example of phase synchronization, we consider two coupled Rössler systems [15]

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + C(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + 0.15y_{1,2}, \\ \dot{z}_{1,2} &= 0.2 + z_{1,2}(x_{1,2} - 10). \end{aligned} \qquad (3)$$

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Here we introduce the parameters $\omega_{1,2} = 1 \pm \Delta\omega$ and C, which govern the frequency mismatch [16] and the strength of coupling, respectively [17]. As the coupling is increased for a fixed mismatch $\Delta\omega$, we observe a transition from a regime, where the phases rotate with different velocities $\phi_1 - \phi_2 \sim \Delta\Omega\, t$, to a synchronous state, where the phase difference does not grow with time $|\phi_1 - \phi_2| <$ const; $\Delta\Omega = 0$. This transition is illustrated in Fig. 1(a). We emphasize that in contrast to the other types of synchronization of chaotic systems [1,2], here the instant fields $x_{1,2}$, $y_{1,2}$, and $z_{1,2}$ do not coincide. Moreover, the correlations between the amplitudes of x_1 and x_2 are pretty small [Fig. 1(b)], although the phases are completely locked and in this respect the motions are highly coherent.

For the Rössler attractor, because of its simple form, the phase can be introduced in a more straightforward way, based on the Poincaré map construction. One can say that the motion from one crossing with a secant surface until the next one corresponds to the phase shift 2π. When we consider coupled chaotic systems, we still can construct partial Poincaré maps, e.g., taking successive maxima of the variables $x_{1,2}$ in the coupled Rössler systems. Partial frequencies are then simply defined as an average number of crossings of the secant surfaces per unit time. According to this approach, the synchronization in coupled Rössler systems simply means that the average numbers of oscillations (number of maxima) per unit time in both systems coincide. The region of synchronization in the plane of parameters "coupling-frequency mismatch," obtained using these partial Poincaré maps, is presented in Fig. 2. Note that it seems to have no threshold: If the frequency mismatch is small $\Delta\omega \to 0$, synchronization appears already for vanishing coupling. This is a particular feature of the Rössler system, where the motion is highly coherent (in the power spectrum a very sharp peak is observed [18]). From the other side, it is possible to synchronize systems with frequency mismatch of more than 20% (see Fig. 2).

The instantaneous phase ϕ, defined through the Hilbert transform (2) provides, of course, additional information on the dynamics of synchronization (see, e.g., the time evolution of $\phi_1 - \phi_2$ for $C = 0.027$ in Fig. 1). We also note that in the case of asymmetric coupling the averaged value of the phase difference can be nonzero. This, e.g., happens in the asymmetric coupling of Rössler systems, where the variable x_1 is driven by y_2 [the first equation in (3) has a form $\dot{x}_1 = -\omega_1 y_1 - z_1 + c y_2$]. Here in the synchronous state the phase difference (both phases are obtained using observables $x_{1,2}$) fluctuates near the mean value $\pi/2$. With the method of partial Poincaré map this particular property is not detectable.

It is remarkable how the phase synchronization manifests itself in the Lyapunov spectrum (Fig. 3). In the absence of coupling, each oscillator has one positive, one negative, and one vanishing Lyapunov exponent. As the coupling is increased, the positive and negative exponents remain, whereas one of the zero exponents becomes negative. This behavior can be explained as follows: Without coupling, the vanishing exponents correspond to the translation along the trajectory, i.e., to the shift of the phase of the oscillator. The coupling produces an "attraction" of the phases such that the phase difference $\phi_1 - \phi_2$ decreases.

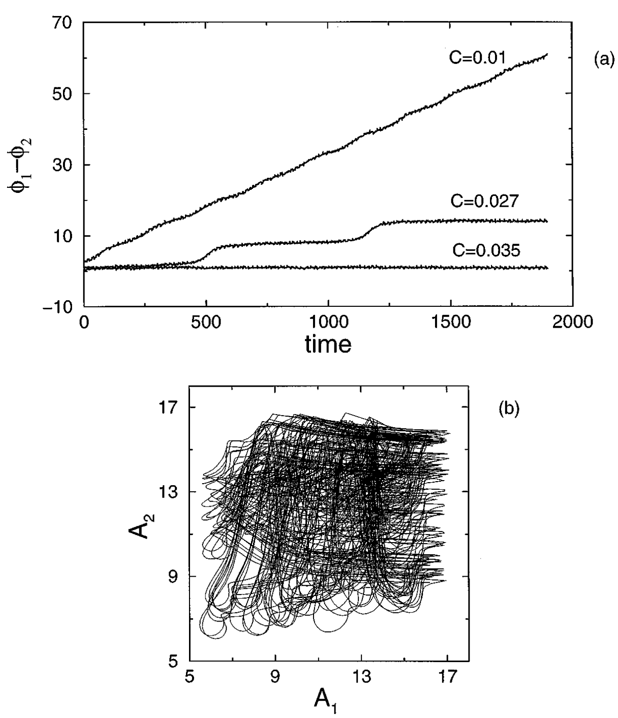

FIG. 1. Phase difference of two coupled Rössler systems [Eq. (3)] versus time for nonsynchronous ($C = 0.01$), nearly synchronous ($C = 0.027$), and synchronous ($C = 0.035$) states (a). In the last case the amplitudes $A_{1,2}$ remain chaotic (b), their cross correlation is less than 0.2. The frequency mismatch is $\Delta\omega = 0.015$.

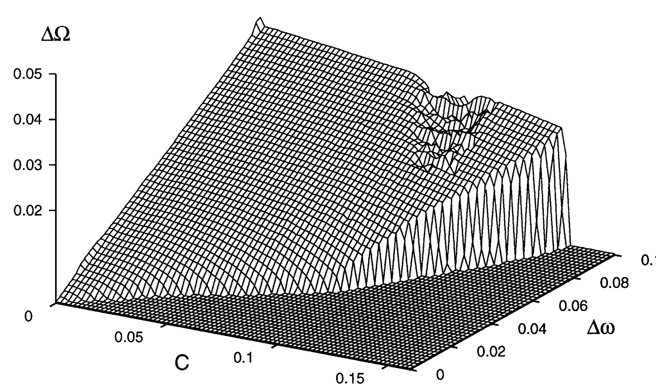

FIG. 2. The mean frequency difference $\Delta\Omega$ for the coupled Rössler systems (3), calculated with the method of partial Poincaré maps, as a function of the coupling C and the frequency mismatch $\Delta\omega$. For C large enough the frequency difference $\Delta\Omega$ is nearly zero; this region of synchronization is completely analogous to the phase-locking domain (the Arnold tongue) for coupled periodic oscillators. For small C there is no synchronization and the phase difference grows with the finite rate $\Delta\Omega$.

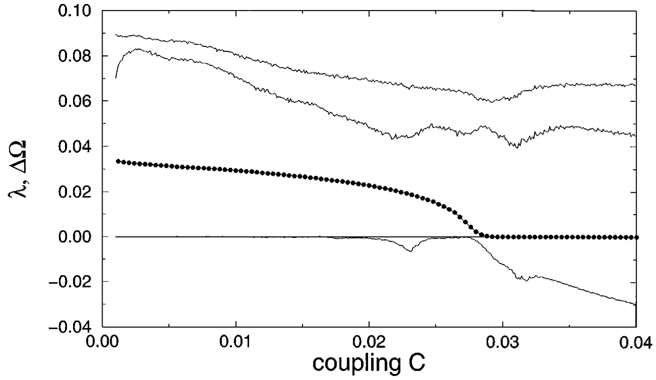

FIG. 3. The four largest Lyapunov exponents, one of which is always zero (lines) and $\Delta\Omega$ (circles) vs coupling C for system (3) with $\Delta\omega = 0.015$.

Thus one of the vanishing exponents becomes negative. For large coupling the attraction is so strong that the phases remain locked.

Qualitatively, the dynamics of the phase of an autonomous chaotic oscillator can be described with the equation (cf. [10,19])

$$\dot{\phi} = \omega + F(A). \tag{4}$$

Here ω is the mean frequency of the oscillations, and the term $F(A)$ accounts for the amplitude dependence of the frequency; the amplitude A is assumed to behave chaotically. For coupled oscillators a generalization of (4) reads

$$\dot{\phi}_{1,2} = \omega_{1,2} + F_{1,2}(A_{1,2}) + \varepsilon G(\phi_{2,1}, \phi_{1,2}). \tag{5}$$

Here G is 2π periodic in each argument function, describing coupling. In the simplest case we can assume that $G(\phi_1, \phi_2) = \sin(\phi_2 - \phi_1)$. Thus for the phase difference $\Delta\phi = \phi_1 - \phi_2$ we get from (5)

$$\frac{d\Delta\phi}{dt} = \omega_1 - \omega_2 - 2\varepsilon \sin(\Delta\phi) + F_1(A_1) - F_2(A_2). \tag{6}$$

This equation is similar to the equation describing phase locking of periodic oscillators in the presence of noise [20]. Here instead of external noisy force we have the term depending on the chaotic amplitudes. In the Rössler attractor the dependence of the frequency on the amplitude is very small, so the effective noise $F_1(A_1) - F_2(A_2)$ in Eq. (6) is negligible, and the dynamics of the phases is very similar to that in the coupled periodic oscillators. This explains the complete phase locking, as well as the absence of the threshold.

It is noteworthy that the phenomenon of phase synchronization is observed even when completely different systems, such as the Rössler oscillator and the Mackey-Glass differential-delay system [21], interact. Here we describe the interaction of the chaotic and the hyperchaotic Rössler oscillators [22]:

$$\begin{aligned}
\dot{x} &= -\omega y - z + C(u - x),\\
\dot{y} &= \omega x + 0.15y,\\
\dot{z} &= 0.2 + z(x - 10),\\
\dot{p} &= -u - v,\\
\dot{u} &= p + 0.25u + w + C(x - u),\\
\dot{v} &= 3 + pv,\\
\dot{w} &= -0.5v + 0.05w.
\end{aligned} \tag{7}$$

Clearly, for the interaction of such different systems there is no hope to observe synchronization in the usual sense [1,2]. However, the phase synchronization occurs in (7), as is demonstrated in Fig. 4. Here we plot the difference between averaged frequencies $\Delta\Omega = \langle \dot{\phi}_1 - \dot{\phi}_2 \rangle$ vs ω (this parameter governs the frequency mismatch), for different coupling strengths C. For this system we have not found a regime with perfect phase locking: Even when $\Delta\Omega \approx 0$, the phase difference $\langle \phi_1 - \phi_2 \rangle$ exhibits a random-type walk and is not constrained [23]. This *weak* phase synchronization can be qualitatively described with the model equation (6) with sufficiently large effective noise $F_1(A_1) - F_2(A_2)$.

In conclusion, we have demonstrated the possibility of phase synchronization of chaotic self-sustained oscillators. In this regime the phases are synchronized, while the amplitudes vary chaotically and are practically uncorrelated. We have described two types of phase synchronization: When interacting chaotic oscillators are highly coherent, the phases are perfectly locked; otherwise, the frequencies are entrained while the phase difference is unbounded. The effect of phase synchronization is also possible when the natural frequencies are in a rational relation (this is relevant for an important physiological problem of interaction of the cardiac and respiratory systems).

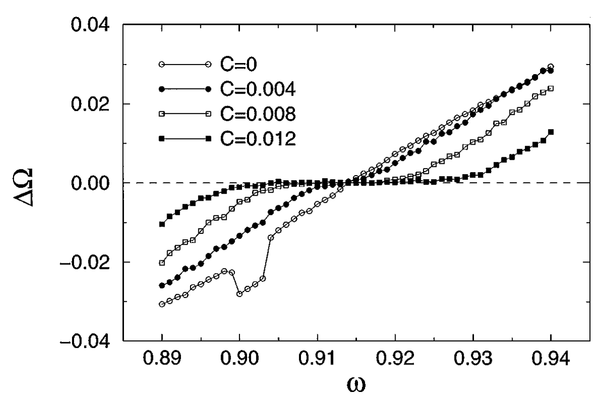

FIG. 4. The mean frequency difference $\Delta\Omega$ in system (7), calculated via Hilbert transform vs ω for several values of the coupling constant C.

We emphasize that the phase synchronization is observed already for extremely weak couplings, and in some cases can have no threshold, contrary to other types of synchronization. This phenomenon is a direct generalization of synchronization of periodic self-sustained oscillators. As the latter, it may find practical applications, in particular, when a coherent summation of outputs of slightly different generators operating in a chaotic regime is necessary. For this purpose, it is sufficient to synchronize phases, while amplitudes can remain uncorrelated. We expect this to be relevant for an important problem of output summation in arrays of semiconductor lasers [24]. For a description of such arrays, as well as of a number of other physical and biological phenomena, one often uses a model of globally coupled oscillators (see, e.g., [25]). Here mutual phase synchronization of individual chaotic states manifests itself as an appearance of a macroscopic mean field [26].

We also mention that the phenomenon of phase synchronization is a characteristic feature of autonomous continuous-time systems, and cannot be observed in discrete-time or periodically forced models. In the latter systems the phases are not free (in the sense of the existence of the zero Lyapunov exponent corresponding to the phase shift) and therefore cannot be adjusted by small coupling.

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[1] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); A. S. Pikovsky, Z. Phys. B **55**, 149 (1984).
[2] L. Pecora and T. Carroll, Phys. Rev. Lett. **64**, 821 (1990); Phys. Rev. A **44**, 2374 (1991).
[3] M. de Sousa Vieira, A. Lichtenberg, and M. Lieberman, Phys. Rev. A **46**, R7359 (1992).
[4] A. S. Pikovsky, Radiophys. Quantum Electron. **27**, 576 (1984).
[5] Y. Kuznetsov, P. Landa, A. Ol'khovoi, and S. Perminov, Sov. Phys. Dokl. **30**, 221 (1985).
[6] I. Blekhman, *Synchronization in Science and Technology* (Nauka, Moscow, 1981) (in Russian) [Synchronization in Science and Technology (ASME Press, New York, 1988)].
[7] C. Hugenii, *Horoloquim Oscilatorium* (Parisiis, France, 1673).
[8] In the presence of noise, a weaker condition for phase locking $|n\phi_1 - m\phi_2| <$ const should be used instead.
[9] P. Landa and N. Tarankova, Radiotechnika i Electronica **21**, 260 (1976) (in Russian); V. S. Anischenko, T. E. Vadivasova, D. E. Postnov, and M. A. Safonova, Int. J. Bif. Chaos **2**, 633 (1992); P. Landa and M. Rosenblum, Appl. Mech. Rev. **46**, 414 (1993); I. Blekhman, P. Landa, and M. Rosenblum, *ibid.* **48**, 733 (1995); *Nonlinear Dynamics: New Theoretical and Applied Results* (Academic Verlag, Berlin, 1995), pp. 17–54.
[10] A. S. Pikovsky, Sov. J. Commun. Technol. Electron. **30**, 85 (1985).
[11] E. F. Stone, Phys. Lett. A **163**, 367 (1992).
[12] D. Gabor, J. IEE London **93**, 429 (1946).
[13] P. Panter, *Modulation, Noise, and Spectral Analysis* (McGraw-Hill, New York, 1965).
[14] For discrete signals given within the bounded time interval, the Hilbert transform can be approximately performed by means of a digital filter, see, e.g., M. J. Smith, *Introduction to Digital Signal Processing* (Wiley, New York, 1992).
[15] O. E. Rössler, Phys. Lett. A **57**, 397 (1976).
[16] Collective coherent behavior in a large array of coupled *identical* Rössler systems has been described in L. Brunnet, H. Chaté, and P. Manneville, Physica (Amsterdam) **78D**, 141 (1994).
[17] One can see that $\omega_{1,2}$ are indeed frequencies of the Rössler system if we rewrite it as $\ddot{y} - a\dot{y} + \omega^2 y = -\omega z$, $\dot{z} + fz = b + z(\dot{y} - ay)/\omega$.
[18] J. Crutchfield *et al.*, Phys. Lett. A **76**, 1 (1980).
[19] J. D. Farmer, Phys. Rev. Lett. **47**, 179 (1981).
[20] R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).
[21] M. Mackey and L. Glass, Science **197**, 287 (1977).
[22] O. E. Rössler, Phys. Lett. A **71**, 155 (1979).
[23] A similar classification was used by H. Daido, Phys. Rev. Lett. **68**, 1073 (1992).
[24] H. G. Winful and L. Rahman, Phys. Rev. Lett. **65**, 1575 (1990); H. G. Winful *et al.*, in *Proceedings of the 1st Experimental Chaos Conference*, edited by S. Vohra, M. Spano, M. Shlesinger, L. Pecora, and W. Ditto (World Scientific, Singapore, 1992), p. 77.
[25] K. Wiesenfeld and P. Hadley, Phys. Rev. Lett. **62**, 1335 (1989); D. Hansel and H. Sompolinsky, Phys. Rev. Lett. **68**, 718 (1992); S. K. Han, C. Kurrer, and Y. Kuramoto, Phys. Rev. Lett. **75**, 3190 (1995).
[26] A. Pikovsky, M. Rosenblum, and J. Kurths (to be published).