A Measurement of the Shape of the Solar Disk: The Solar Quadrupole Moment, the Solar Octopole Moment, and the Advance of Perihelion of the Planet Mercury

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The Solar Disk Sextant experiment has measured the solar angular diameter for a variety of solar latitudes. Combined with solar surface angular rotation data, the solar quadrupole moment J_2 and the solar octopole moment J_4 have been derived first by assuming constant internal angular rotation on cylinders and then by assuming constant internal angular rotation on cones. We have derived values of 1.8×10^{-7} for J_2 and 9.8×10^{-7} for J_4. We conclude with a discussion of errors and address the prediction of general relativity for the rate of advance of perihelion of the planet Mercury.

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The solar oblateness is defined in terms of the difference in radius between the solar pole and the solar equator,

$$(R_{\text{equator}} - R_{\text{pole}})/R_\odot = \Delta R/R_\odot\,. \tag{1}$$

By assuming that the solar internal rotation Ω is constant on cylinders (i.e., $\partial\Omega/\partial z = 0$), the oblateness can be related directly to the solar quadrupole moment J_2 through a series of simple assumptions [1–3]. The quadrupole moment contributes a potentially non-negligible component to the advance of perihelion of the planet Mercury [4], one of the "classic" tests of general relativity. The agreement of the prediction of general relativity with the observed value of 42.98 ± 0.04 arc sec century^{-1} [4,5] was hailed as a triumph for the theory even though the quadrupole contribution to the perihelion shift was not evaluated easily. Dicke and Goldenberg [6] called the validity of general relativity into question by reporting a measurement of the solar oblateness so large that the corresponding perihelion shift would result in an unacceptable disagreement between general relativity and observation. While subsequent measurements of the oblateness have yielded values more consistent with the prediction of general relativity [7,8], the value of the quadrupole moment has remained elusive. Using solar oscillation data, Brown *et al.* [9] were able to evaluate $\Omega(r,\theta)$ and derived a value of 1.7×10^{-7} for J_2, but did not provide reliable estimates of the uncertainties. As advocated by Ulrich and Hawkins [10,11], a precise determination of J_2 and higher moments requires a knowledge of both the mass and angular velocity distribution within the Sun. Despite advances in helioseismology, neither quantity is well-agreed upon, so we have addressed the evaluation of J_2 using more traditional means and the latest results from the Solar Disk Sextant (SDS) experiment.

The SDS is a balloon-borne experiment flown over periods between 6 and 10 h during which the solar angular diameter is measured at a variety of orientations [12,13]. The experiment employs linear charge coupled devices (CCD's) which provide runs of intensity versus angular radius at seven different orientations perpendicular to the solar limb; the SDS records eighteen sets of limb profiles every 3 sec. The edge of the solar limb is determined by matching a template intensity profile function to the observed profile. Hill *et al.* [7,14] have demonstrated that the finite Fourier transform definition (FFTD) permits the determination of the solar edge without resorting to an explicit expression for the template intensity function. The reduction of the SDS data employs a similar technique [15,16]. The CCD images allow for the easy detection of sunspots, faculae, flares, and other solar surface phenomena that show up as irregularities in the usually smooth limb profiles. During a flight, over 10^5 images are taken, but, after removing all images contaminated by surface effects or excessive telescope drift, only two-thirds are used in determining the size and shape of the solar disk. Kuhn and Libbrecht [17], however, have reported that the solar photospheric temperature and temperature gradient may be functions of the solar activity cycle. If so, then the use of the FFTD may introduce a systematic error in our results. While we are in the process of using the SDS limb profiles to quantify this effect, for now, we can only acknowledge nonfacular temperature variations as a potential source of uncertainty. The SDS experiment was partially motivated by the suggestion of Dicke, Kuhn, and Libbrecht [18] that the magnitude of the oblateness might be a function of the solar cycle. Based upon flights in 1992 and 1994, the measured oblateness was $(9.17 \pm 1.25) \times 10^{-6}$ and $(8.77 \pm 0.99) \times 10^{-6}$ [19–21], indicating little or no variation and remaining fully consistent with Hill's value of $(9.6 \pm 6.5) \times 10^{-6}$ [7,8].

The theory of rotating stars has revealed that for a star with internal rotation such that $\partial\Omega/\partial z = 0$, the effective gravity can be derived from an effective potential, Ψ, such that surfaces of constant pressure, density, and Ψ are coincident [22]. This idealized case can yield an evaluation of J_2 from the oblateness. Given a measure of the surface radius

$$R_{\text{surf}}(\theta) = R_\odot\left[1 + \sum_{n=2}^{\text{even } n} r_n P_n(\cos\theta)\right], \tag{2}$$

where the functions P_n are Legendre polynomials, this technique can be extended to higher moments yielding expressions of the form

$$J_n = f_n(\Omega^2) - r_n \,. \tag{3}$$

Ulrich *et al.* [23] have examined the solar surface velocity field using 20 years of data from the Mt. Wilson Observatory and have fit the surface angular velocity as

$$\Omega_{\rm surf}(\psi) = A + B[\sin^2(\psi) + (1.022)\sin^4(\psi)], \tag{4}$$

where ψ represents latitude. Averaging their coefficients between 1979 and 1988 yielded a value of 2.840 μrad sec^{-1} for A and -0.400 μrad sec^{-1} for B. Assuming that the surface rotation could be used to represent the internal rotation, $\Omega_{\rm surf}(\psi)$ was rewritten as $\Omega(\varpi)$ and then used in Eq. (3). The coefficients r_n were evaluated from the SDS data which are displayed in Fig. 1. For the quadrupole term

$$f_2(\Omega^2) = -5.625 \times 10^{-6}, \qquad r_2 = -5.810 \times 10^{-6}, \qquad \text{and so } J_2 = 1.84 \times 10^{-7}.$$

For the octopole term

$$f_4(\Omega^2) = 5.66 \times 10^{-7}, \qquad r_4 = -4.17 \times 10^{-7}, \qquad \text{and so } J_4 = 9.83 \times 10^{-7}.$$

Since the SDS data were not of sufficient quality to extract coefficients beyond r_4, only crude estimates could be made of higher moments up to $n = 10$,

$$J_6 \sim f_6(\Omega^2) = 4 \times 10^{-8}, \qquad J_8 \sim f_8(\Omega^2) = -4 \times 10^{-9}, \qquad J_{10} \sim f_{10}(\Omega^2) = -2 \times 10^{-10}.$$

The large value of the octopole term was unusual. For the Earth, the ratio $|J_4/J_2^2|$ is close to unity [24]. Using a different technique, Ulrich and Hawkins [10,11] first suggested that for the Sun $|J_4| \gg |J_2^2|$, and this was certainly the case for the SDS results.

Although idealized solar structure has been used previously to extract moments from the shape of the solar limb, a thorough discussion of the errors and uncertainties in the results has been lacking. If general relativity properly describes the advance of perihelion of Mercury, then 42.98 ± 0.04 arc sec century^{-1} corresponds to a quadrupole moment of $(2.3 \pm 3.1) \times 10^{-7}$. How significantly different is this evaluation of J_2 from our result? We considered three sources of uncertainty. First, the data in Fig. 1 revealed that r_2 was equal to $(-5.810 \pm 0.400) \times 10^{-6}$. Although the relative uncertainty in r_2 was small, the similarity with the f_2 term in Eq. (3) translated into an enormous relative error in J_2: $\pm 4.0 \times 10^{-7}$. Since the SDS results represent the most accurate determination of the shape of the solar disk, uncertainties of greater magnitude must exist for all similar determinations of J_2. For the octopole term, r_4 was equal to $(-4.17 \pm 4.59) \times 10^{-7}$, or J_4 was properly evaluated as $(9.8 \pm 4.6) \times 10^{-7}$. Ironically, the determination of the octopole moment might appear to be a more "robust" result than the determination of the quadrupole moment even though the relative uncertainty in r_2 was only 7%, while for r_4 it was greater than 100%.

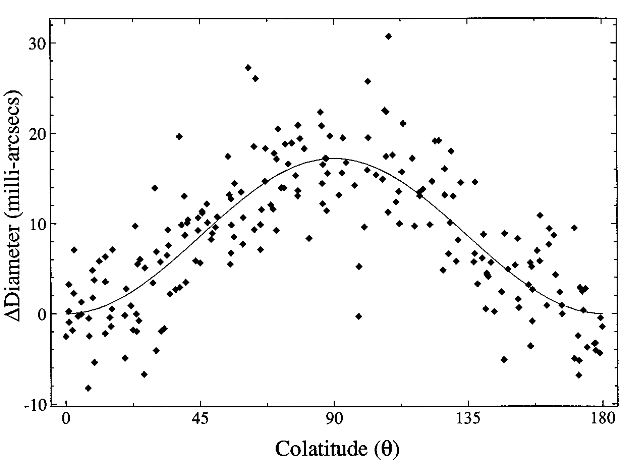

FIG. 1. The variation of the solar diameter with colatitude as provided by the SDS experiment. The fit to the data is $17.249[1 - \cos(2\theta)]$.

Second, the coefficients in the surface rotation curve in Eq. (2) were not without their own uncertainty. By averaging the data of Ulrich *et al.* [23] from 1979 to 1988, we evaluated the coefficient A as 2.840 ± 0.025 μrad sec^{-1}, which contributed an uncertainty of $\pm 1.1 \times 10^{-7}$ to J_2 (the effect on J_4 was negligible as were the effects of uncertainties in B on either of the moments). While the large number of observations of the surface rotation might have allowed us to lower the uncertainty below 0.025, we did not do so for two reasons. First, all SDS observations were made on two days, two years apart rather than continuously over a period of many years. Second, observations of the solar surface rotation were not available for the period from 1992 to 1994. Both these facts required a more generous estimate of potential uncertainties. For a summary of recent evaluations of $\Omega_{\rm surf}(\psi)$, see Stix [3] or, for a more penetrating review of both methods and uncertainties, see Schröter [25].

Finally, recent work on the inversion of p modes has revealed that the solar internal rotation may be more complex than constant angular rotation on cylinders [9,26] with constant rotation on cones being a better approximation [i.e., $\Omega(\theta)$ [27,28]]. In order to investigate this possibility, we started with the momentum balance equation [10,29] and constructed a model of the surface layers of the Sun matching as closely as possible the latest results from solar atmosphere models [30]. We assumed that the shape of the solar surface as measured by the SDS corresponded to a surface of constant density. Initially, we assumed that surfaces of constant density and pressure were coincident and allowed the model to relax to an equilibrium state. This process yielded values for J_2 through J_{10} which were identical to those provided earlier. In the next step, the surfaces of constant pressure were made to deviate from the surface of constant density by an amount derived by assuming constant rotation on cones throughout the solar interior. The form of $\partial\Omega(\theta)/\partial z$ was such that the degree of deviation was small: Isobaric and isopycnic surfaces coincident at the pole were offset by a distance of about 1 m at the equator. This offset altered the values of the moments only slightly: 1.86×10^{-7} for the quadrupole term and 9.80×10^{-7} for the octopole term.

The purpose of this paper was to determine as accurately as possible the solar quadrupole and octopole moments. The SDS results provided a value for J_2 of 1.8×10^{-7} and a value for J_4 of 9.8×10^{-7}. The uncertainties are such that this evaluation of J_2 is consistent with the prediction of general relativity. The magnitude of J_4 is greater than the quadrupole term regardless of the observational and theoretical uncertainties; this result warrants further investigation. Finally, by evaluating all of the r_n coefficients in Eq. (2), we found that $R_{\rm surf} = R_\odot$ at $\theta \approx 53°$ or $\psi \approx 37°$ yielding our definition of $R_\odot$. Future plans for the SDS include continued yearly or biyearly flights and a proposed Arctic flight lasting several weeks. The amount and quality of data from the latter flight should allow for the detection of low-frequency p modes, g modes and provide a more precise evaluation of the coefficients r_2 and r_4. Simultaneous observations of the solar surface rotation would ultimately result in a more stringent test of general relativity.

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