Directed Compact Lattice Animals, Restricted Partitions of an Integer, and the Infinite-State Potts Model

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We consider a directed compact site lattice animal problem on the d-dimensional hypercubic lattice, and establish its equivalence with (i) the infinite-state Potts model and (ii) the enumeration of $(d-1)$-dimensional restricted partitions of an integer. The directed compact lattice animal problem is solved exactly in $d = 2, 3$ using known solutions of the enumeration problem. The maximum number of lattice animals of size n grows as $\exp(cn^{(d-1)/d})$. Also, the infinite-state Potts model solution leads to a conjectured limiting form for the generating function of restricted partitions for $d > 3$, the latter an unsolved problem in number theory.

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An intriguing aspect of lattice statistics is that seemingly totally different problems are sometimes related to each other, and that the solution of one problem can often be used to solve other outstanding unsolved problems. An example is the $d = 2$ directed lattice site animals solved by Dhar [1] who used Baxter's exact solution of a hard-square lattice gas model [2,3] to deduce its solution. In this Letter we consider a directed *compact* site lattice animal problem in d dimensions, and show that it is related to (i) the infinite-state Potts model in d dimensions and (ii) the enumeration of $(d-1)$-dimensional restricted partitions of an integer. The known solutions of restricted partitions in two and three dimensions [4,5] now solve the corresponding compact lattice animal problems, and, similarly, the established solution of the infinite-state Potts model [6] leads to a conjectured limiting form for the generating function of restricted partitions for $d > 3$, which is an outstanding unsolved problem in number theory. For clarity of presentation, we present details of discussions for $d = 2$. Considerations in higher dimensions are similar, and relevant results will be given.

Directed compact lattice animals and restricted partitions of an integer.—Starting from the origin {1,1} of an $L_1 \times L_2$ simple quartic lattice $\mathcal{L}$ whose columns and rows are numbered, respectively, by $i = 1, \ldots, L_1$ and $j = 1, \ldots, L_2$, a site animal grows in the directions of increasing i and j. In contrast to the previously considered directed animal problem [1] for which a site $\{i, j\}$ can be occupied when either the site $\{i-1, j\}$ or the site $\{i, j-1\}$ is occupied, we introduce a more restricted growth rule. Our rule is that a site $\{i, j\}$ can be occupied only when both $\{i-1, j\}$ and $\{i, j-1\}$ are occupied. (When applying the growth rule, sites with coordinates $i = 0$ or $j = 0$ are regarded as being occupied.) Compared to the usual directed lattice animals [1], the present model generates compact animals since it excludes configurations with unoccupied interior sites. In addition, we keep L_1, L_2 finite, so that there exists a maximum animal size of $L_1 L_2$.

Let $A_n(L_1, L_2)$ be the number of distinct n-site compact animals that can grow on $\mathcal{L}$. In considering animal problems, one is primarily interested in finding the asymptotic behavior $A_n(L_1, L_2)$ for large n. It is clear that by keeping L_1, L_2 finite the asymptotic behavior will depend on the relative magnitudes of n, L_1, L_2. The study of enumerations is facilitated by the use of generating functions. In the present case we define the generating function

$$G(L_1, L_2; t) = 1 + \sum_{n=1}^{L_1 L_2} A_n(L_1, L_2) t^n. \tag{1}$$

For example, the generating function for the 3×3 lattice is

$$G(3,3;t) = 1 + t + 2t^2 + 3t^3 + 3t^4 + 3t^5 + 3t^6 + 2t^7 + t^8 + t^9. \tag{2}$$

We observe that $A_n(L_1, L_2)$ reaches a maximum at $n \sim L_1 L_2 / 2$.

Let h_i, $i = 1, 2, \ldots, L_1$, be the number of occupied sites in the ith column of $\mathcal{L}$. One observes that our growth rule implies the restriction

$$L_2 \geq h_1 \geq h_2 \geq \cdots \geq h_{L_1} \geq 0. \tag{3}$$

In addition, one has the (one-dimensional) sum rule

$$\sum_{i=1}^{L_1} h_i = n, \tag{4}$$

where n is the animal size. It is convenient to regard (4) as specifying the partitions of a positive integer n into sums of integers h_i, and the condition (3) ensures that all partitions are distinct. Then $A_n(L_1, L_2)$ is precisely the number of distinct ways that an integer n is partitioned into at most L_1 parts with each part less than or equal to L_2. This leads to a classic restricted *partitio numerorum* problem dating back to Gauss [4]. Particularly, the generating function (1) can be evaluated in a closed form [5,7]

$$G(L_1, L_2; t) = (t)_{L_1+L_2}/(t)_{L_1}(t)_{L_2}, \qquad (5)$$

where $(t)_L \equiv \prod_{p=1}^{L}(1 - t^p)$. Note that, despite its appearance, all zeros in the denominator are canceled and (5) is a true polynomial in t as shown in (2). The generating function (1) is known as the Gaussian polynomial or the "q coefficient."

There are $L_1L_2 + 1$ terms in (1) whose coefficients satisfy the sum rule

$$1 + \sum_{n=1}^{L_1L_2} A_n(L_1, L_2) = \binom{L_1 + L_2}{L_1} \qquad (6)$$

and the symmetry

$$G(L_1, L_2; t) = t^{L_1L_2} G(L_1, L_2; t^{-1}). \qquad (7)$$

While these two properties are relatively easy to prove [5], the Gaussian polynomial possesses a unimodal property, namely, $A_{n-1}(L_1, L_2) < A_n(L_1, L_2)$ for $n \le L_1L_2/2$, which is very deep. A combinatorial proof of this unimodal property appeared only very recently [8].

The Gaussian polynomial can be inverted by the Cauchy integral to yield

$$A_n(L_1, L_2) = \frac{1}{2\pi i}\oint \frac{1}{z^{n+1}} G(L_1, L_2; z)\, dz, \qquad (8)$$

where the integration is taken over a contour inside $|z| = 1$, enclosing the origin. The asymptotic behavior of $A_n(L_1, L_2)$ for large n can be deduced by applying saddle point analyses to (8). For $n < \min\{L_1, L_2\}$, the rows and columns of $\mathcal{L}$ are never fully filled so that the partition of n is actually without restrictions. Then, the classic analysis of (8) by Rademacher [9] with $G(L_1, L_2; z)$ effectively replaced by the Eulerian product $(t)_\infty^{-1}$ yields the celebrated Hardy-Ramanujan [10] result

$$A_n(L_1, L_2) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \quad n < \min\{L_1, L_2\}. \qquad (9)$$

Clearly, the asymptotic behavior of $A_n(L_1, L_2)$ changes as n increases, and the partition of n becomes more restricted. When $A_n(L_1, L_2)$ assumes its maximum value at $n = L_1L_2/2$ (the unimodal property), the leading contribution can be obtained by observing that the left-hand side of (6) consists of $L_1L_2 + 1$ *positive* terms of which the largest term is of the order of $e^{c\sqrt{n}}$, where c is a constant. It follows that to the leading order the largest term is well approximated by the sum $\binom{L_1+L_2}{L_1}$. This leads to the asymptotic behavior

$$A_n(L_1, L_2) \propto e^{c(\alpha)\sqrt{n}}, \quad n \sim L_1L_2/2, \qquad (10)$$

where

$$c(\alpha) = \sqrt{2}\left[\sqrt{\frac{1}{\alpha}}\ln(1+\alpha) + \sqrt{\alpha}\ln\left(1 + \frac{1}{\alpha}\right)\right] = c(\alpha^{-1}),$$
$$\alpha = L_1/L_2. \qquad (11)$$

Assuming a Gaussian distribution for $A_n(L, L)$ near its center $n = L^2/2$ ($\alpha = 1$), it can be shown [11] that

$$A_n(L, L) \sim \left(\frac{\sqrt{3}}{2\pi n}\right) 2^{2\sqrt{2n}}, \quad n \sim L^2/2. \qquad (12)$$

It therefore appears safe to conclude that the asymptotic behavior of $A_n(L_1, L_2)$ assumes the universal form of $n^{-1}e^{c\sqrt{n}}$, where c is a constant which decreases with increasing n. The initial value of c is $\pi\sqrt{2/3} = 2.5651\ldots$ for $n < \{L_1, L_2\}$. Its value decreases to $c(\alpha) < c(1) = 2\sqrt{2}\ln 2 = 1.9605\ldots$ for $n \sim L_1L_2/2$ and eventually to 0 for $n \sim L_1L_2$ when the lattice is fully occupied. This is to be compared with the asymptotic behavior $3^n n^{-3/2}$ of the usual directed animals [1].

Equivalence with an infinite-state Potts model.— Consider the standard Potts model with reduced nearest-neighbor interactions K on an $L_1 \times L_2$ simple quartic lattice with the special boundary conditions shown in Fig. 1. It has been recently conjectured [12] that zeros of the Potts partition function on this (self-dual) lattice lie on the unit circle $|x| = 1$ in the $\mathrm{Re}(x) > 0$ half plane for all L_1 and L_2, where $x = (e^K - 1)/q^{1/2}$. As a by-product of our analysis, we shall establish this conjecture in the $q = \infty$ limit.

The high-temperature expansion of the Potts partition function assumes the form [13]

$$Z_{L_1,L_2}(q, x) = \sum_{\text{bond config}} x^b q^{n+b/2}, \qquad (13)$$

where the summation is taken over all $2^{2L_1L_2}$ bond-covering configurations of the lattice; b and n are, respectively, the numbers of bonds and connected clusters (including isolated points) of each configuration. In the large q limit, the leading terms in (13) are of the order of $q^{L_1L_2+1}$. This factor can be achieved by taking, for example, the fully covered bond configuration of $n = 1, b = 2L_1L_2$ with the weight $q^{L_1L_2+1}x^{2L_1L_2}$. It is then convenient to introduce the reduced ($q = \infty$) Potts partition function

$$Y_{L_1,L_2}(x) \equiv \lim_{g\to\infty} q^{-(L_1L_2+1)} Z_{L_1,L_2}(q, x). \qquad (14)$$

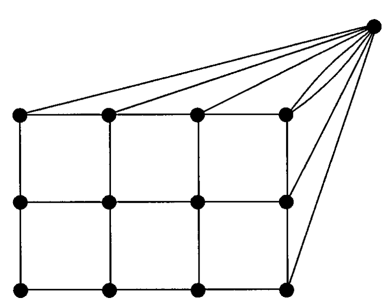

FIG. 1. A 4×3 lattice with 13 sites and 24 edges.

We now establish the identity

$$Y_{L_1,L_2}(x) = x^{2L_1L_2}G(L_1,L_2;x^{-2}) = G(L_1,L_2;x^2)\,. \tag{15}$$

To prove (15), we consider the generation of Y_{L_1,L_2} from a systematic removal of bonds starting from the fully covered configuration. Generally, to hold the number $n + b/2$ constant, the minimum one can do is to decrease b by 2 while increasing n by 1. Thus, one always looks for sites connected to exactly two neighboring sites. Starting from the fully covered configuration, one observes from Fig. 1 that there is only one such site, namely, the site {1,1} at the lower-left corner, which is connected to the two sites {1,2} and {2,1}. Removing the two bonds connecting to {1,1}, one creates a configuration of $n = 2$ and $b = 2L_1L_2 - 2$ with the weight $x^b = x^{2L_1L_2}x^{-2}$. We regard the now isolated site {1,1} as a one-site animal.

Repeating this procedure, one next looks for the one-site animal configuration sites which are connected to exactly two neighboring sites. There are now two such sites, namely, {1,2} and {2,1}. By removing the two bonds connected to either of the two sites, one finds the next term in the reduced partition function having $n = 3$, $b = 2L_1L_2 - 4$ and the weight $2x^b = 2x^{2L_1L_2}x^{-4}$. The resulting configurations now have two isolated sites which can be regarded as two-site animals. Continuing in this fashion, it is recognized that the process of creating isolating sites (by removing two bonds at a time) follows precisely our rule of growing directed animals on $\mathcal{L}$. It follows that we have established the first line of (15). The second line of (15) now follows from (7). It should be pointed out that our proof of (15) works equally well for the Potts model with anisotropic reduced interactions K_1 and K_2. The reduced partition function is again given by (15) but with the replacement of x^2 by x_1x_2, where $x_i = (e^{K_i} - 1)/\sqrt{q}$. We have also established that all zeros of $Z_{L_1,L_2}(\infty, x)$ are on the unit circle $|x| = 1$, verifying a conjecture of [12] in the $q = \infty$ limit.

Since all zeros of the Gaussian polynomial are on the unit circle $|x| = 1$, one can introduce a per-site reduced free energy for the $q = \infty$ Potts model as [12,14]

$$f(x^2) \equiv \lim_{L_1,L_2\to\infty}(L_1L_2)^{-1}\ln G(L_1,L_2;x^2) = \int_{-\pi}^{\pi} g(\theta)\ln(e^{i\theta} - x^2)\,d\theta\,, \tag{16}$$

where $L_1L_2g(\theta)$ is the density of zeros of $G(L_1,L_2;x^2)$ on the unit circle in the complex x^2 plane. To determine $g(\theta)$, we note that the zeros of $(t)_L = \prod_{p=1}^{L}(1 - t^p)$ are at $e^{i\theta_{\ell p}}$, where

$$\theta_{\ell p} = 2\pi\ell/p, \quad p = 1,2,\dots,L; \quad \ell = 1,2,\dots,p\,. \tag{17}$$

This implies that, as p ranges from 1 to L, the number of zeros on an arc of the unit circle $|t| = 1$ between the real axis and any angle θ is equal to $\theta L^2/4\pi$, the area of the right triangle with perpendicular sides L and $\theta L/2\pi$. It follows that the density of the zeros of $(t)_L$ on the circle $|t| = 1$ is a constant equal to

$$H_L(\theta) = L^2/4\pi\,. \tag{18}$$

Consequently, from (5), the density of zeros of $G(L_1,L_2;t)$ in the complex-t plane is also a constant and equal to $H_{L_1+L_2}(\theta) - H_{L_1}(\theta) - H_{L_2}(\theta) = L_1L_2/2\pi$. This leads to $g(\theta) = 1/2\pi$, and the integral (16) can be evaluated, yielding

$$f(x^2) = \begin{cases} \ln|x^2|, & |x| > 1\,, \\ 0\,, & |x| < 1\,, \end{cases} \tag{19}$$

confirming the known first-order transition of the infinite-state Potts model [6,15].

Results in d dimensions.—The above consideration can be extended to d dimensions [16]. Define directed compact lattice animals which grow from the origin of a d-dimensional hypercubic lattice $\mathcal{L}$ of size $L_1 \times \cdots \times L_d$ in the d-positive directions subject to the constraint that a site $\{i_1, i_2, \dots, i_d\}$ can be occupied only when the d sites $\{i_1, i_2, \dots, i_s - 1, \dots, i_d\}$, $s = 1,2,\dots,d$ are all occupied. Let $A_n(L_1,L_2,\dots,L_d)$ be the number of directed compact animals of size n. Then $A_n(L_1,L_2,\dots,L_d)$ is the number of distinct partitions of a positive integer n into sums of nonnegative integers $m(n_1,n_2,\dots,n_{d-1})$ associated with vertices $\{n_1,n_2,\dots,n_{d-1}\}$ of a $(d-1)$-dimensional hypercubic lattice, or, explicitly,

$$n = \sum_{n_1=1}^{L_1}\sum_{n_2=1}^{L_2}\cdots\sum_{n_{d-1}=1}^{L_{d-1}} m(n_1,\dots,n_{d-1}), \quad m(n_1,\dots,n_{d-1}) > 0\,, \tag{20}$$

such that

$$0 \le m(n_1',n_2',\dots,n_{d-1}') \le m(n_1,n_2,\dots,n_{d-1}) \le L_d\,, \tag{21}$$

whenever $n_1 \le n_1', n_2 \le n_2', \dots, n_{d-1} \le n_{d-1}'$. This defines a $(d-1)$-dimensional restricted partition [5].

In a similar fashion one defines the generating function

$$G(L_1,L_2,\dots,L_d;t) = 1 + \sum_{n=1}^{L_1L_2\cdots L_d} A_n(L_1,L_2,\dots,L_d)t^n, \tag{22}$$

and, analogous to (15), establishes [16] that the generating function (22) is precisely the reduced partition function of the infinite-state Potts model [17] on $\mathcal{L}$, provided that one identifies $t = x^d$ and $x = (e^K - 1)/q^{1/d}$.

But explicit expressions of the generating function (22) are known only for $d = 2$ and $d = 3$. For $d = 2$ it is given by (5), and for $d = 3$ it is [5,7]

$$G(L_1,L_2,L_3;t) = \frac{[t]_{L_1+L_2+L_3-1}[t]_{L_1-1}[t]_{L_2-1}[t]_{L_3-1}}{[t]_{L_1+L_2-1}[t]_{L_2+L_3-1}[t]_{L_1+L_3-1}}\,, \tag{23}$$

where

$$[t]_L \equiv \prod_{p=1}^{L} (t)_p, \quad (t)_L \equiv \prod_{p=1}^{L} (1 - t^p). \tag{24}$$

We observe from (23) that zeros of $G(L_1, L_2, L_3; t)$ are on the unit circle $|t| = 1$ with a uniform density $L_1L_2L_3/2\pi$, leading again to the per-site reduced free energy (19) with x^2 replaced by x^3 in agreement with the known solution [6]. In addition, the asymptotic behavior of the largest $A_n(L_1, L_2, L_3)$, which we expect as in $d = 2$ to occur at $n \sim L_1L_2L_3/2$ and is the same as that of $G(L_1, L_2, L_3; 1)$, is [18]

$$A_n(L_1, L_2, L_3) \propto \exp[c(\alpha_1, \alpha_2, \alpha_3)n^{2/3}], \quad n \sim L_1L_2L_3/2, \tag{25}$$

where

$$c(\alpha_1, \alpha_2, \alpha_3) = 2^{-1/3}\left[\left(\frac{\alpha_1}{\alpha_2}\right)^{1/3} + \left(\frac{\alpha_2}{\alpha_3}\right)^{1/3} + \left(\frac{\alpha_3}{\alpha_1}\right)^{1/3}\right]t(\alpha_1, \alpha_2, \alpha_3),$$

$$t(\alpha_1, \alpha_2, \alpha_3) = (x_1x_2 + x_2x_3 + x_3x_1)^{-1} \sum_{i=1}^{3} [x_i^2 \ln x_i - (1 - x_i)^2 \ln(1 - x_i)], \tag{26}$$

with $x_i = (1 + \alpha_j + 1/\alpha_k)^{-1}, \alpha_i = L_j/L_k, i, j, k$ in cyclic order of 1,2,3. Particularly, for $L_1 = L_2 = L_3 = L$, one has $c(1,1,1) = 2^{2/3}(9\ln\sqrt{3} - 3\ln 4) = 1.245\,907$. Expressions (10) and (25) suggest the asymptotic behavior

$$A_n(L_1, L_2, \ldots, L_d) \propto \exp(cn^{(d-1)/d}), \quad n \sim L_1L_2\cdots L_d/2 \tag{27}$$

for general d. However, the problem of $(d-1)$-dimensional restricted partitions of a positive integer for $d > 3$ is an outstanding unsolved problem in number theory. In fact, it can be verified by considering a $2 \times 2 \times 2 \times 2$ lattice that zeros of the generating function (22) are no longer on the unit circle. On the other hand, the $q = \infty$ Potts model is known to have a first-order transition at $x \equiv (e^K - 1)/q^{1/d} = 1$ [6]. Our results in $d = 2, 3$ then suggest that the generating function (22) can be evaluated in the thermodynamic limit as

$$\lim_{L_1,\ldots,L_d\to\infty} (L_1\cdots L_d)^{-1} \ln G(L_1, \ldots, L_d; t) = \begin{cases} \ln|t|, & |t| > 1, \\ 0, & |t| < 1. \end{cases} \tag{28}$$

We conjecture that (27) and (28) hold for any $d > 1$.

Finally, we remark that in deducing (28) we have assumed the special boundary condition [17] and interchanged the $q \to \infty$ and the thermodynamic limits. While the interchange of the two limits is a subtle matter, it can be explicitly verified in the $d = 1$ solution that the two limits indeed commute under the boundary conditions of [17].

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