Linear ac Response of Thin Superconductors during Flux Creep

Ernst Helmut Brandt
Max Planck Institute für Metallforschung, Institut für Physik, D-70506 Stuttgart, Germany

Alexander Gurevich
Applied Superconductivity Center, University of Wisconsin, Madison, Wisconsin 53706
(Received 20 October 1995)

The linear magnetic susceptibility $\chi(\omega)$ of superconducting strips and disks in a transverse magnetic field is calculated in the flux-creep regime. It is shown that $\chi(\omega) = \chi' - i\chi''$ for $\omega \gg 1/t$ is *universal,* independent of temperature, dc field, and material parameters, depending only on the sample shape, ac frequency $\omega/2\pi$, and time t elapsed after creep has started. Qualitatively, $\chi(\omega)$ can be obtained from the $\chi(\omega)$ of metallic conductors by replacing the Ohmic relaxation time by t. At $\omega t \gg 1$, which may apply down to rather low frequencies, the dissipative flux-creep state exhibits a nearly Meissner-like response with $\chi' = -1 + 0.40/\omega t$ and $\chi'' = 0.25 \ln(29\omega t)/\omega t$ for disks.

PACS numbers: 74.60.Ge, 74.25.Ha

The theory of the ac susceptibility $\chi(\omega)$ of type-II superconductors, and especially of high-T_c materials exhibiting significant flux creep, is still incomplete. To obtain larger signals, $\chi(\omega)$ is usually measured on thin specimens in a transverse ac magnetic field B_a, whereas the common analysis of such data often assumes long specimens in longitudinal field. Only recently theories became available for the Bean critical state of thin circular disks [1] and strips [2,3], and of rectangular disks [4] in a transverse field. These theories were confirmed also by magneto-optic observations [5]. Besides this *nonlinear* quasistatic response, the *linear* response of thin conductors with arbitrary linear complex and frequency dependent resistivity caused, e.g., by thermally activated flux-line motion, was calculated in longitudinal [6–8] and transverse [4,8–12] geometries. Thus the nonlinear quasistatic response far below the irreversibility line (see Ref. [13] for a comparative review) and the linear response above the irreversibility line in principle are known for the relevant geometries. However, one still lacks theoretical understanding of ac experiments performed in the flux-creep regime below the irreversibility line, where the dynamic response changes with time and can be both linear and nonlinear.

A universality of flux creep well above the magnetic field of full flux penetration, H_p, was recently demonstrated for longitudinal [14] and transverse [15] geometries. Namely, if the applied dc magnetic field H_a is held constant at times $t > 0$, then, after some transient time depending on the previous ramp rate dB_a/dt, flux creep induces an electric field

$$E_0(\eta, t) = \alpha f(\eta)/t, \tag{1}$$

where $f(\eta)$ is a universal function, and the prefactor $\alpha = ad\mu_0 j_\rho/2\pi$ depends on the specimen. For strips of width $2a$ and disks of radius a and of thickness $d \ll a$ in a transverse field, the profiles $f_s(\eta)$ and $f_d(\eta)$ are plotted in Fig. 1 versus the reduced spatial variable $\eta = x/a$ or $\eta = r/a$, respectively. The two constitutive laws entering this theory are $B = \mu_0 H$ for $H \gg H_p \approx dj_c/\pi$ and a sufficiently nonlinear dependence of $E(j) = E_c \exp[-U(j)/T]$ on the current density j in the thermally activated flux-creep state for which the differential resistivity $\rho(E) = \partial E(j)/\partial j$ is given by

$$\rho(E) \approx E/j_\rho\,. \tag{2}$$

Here $j_\rho(T,H) \approx dj/d\ln t$ is the observed flux-creep rate. For $E(j) = E_c \exp[-U(j)/T]$ one has $j_\rho =$

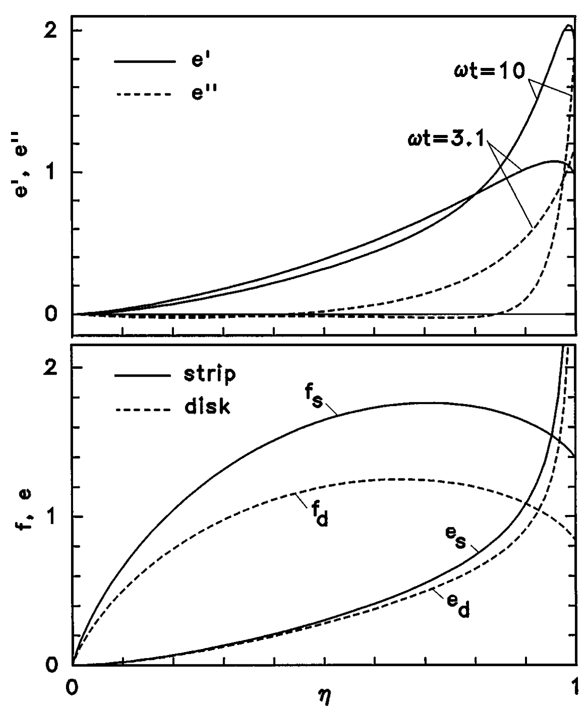

FIG. 1. The universal profiles of the electric field during flux creep in strips (f_s, e_s) and disks (f_d, e_d) versus $\eta = x/a = r/a$. $f(\eta)$ (7) is the usual creep profile, to which a complex ac perturbation $e(\eta, \omega t)$ (17) is superimposed. The upper plot shows the real and imaginary parts of $e = e' + ie''$ for a disk at $\omega t = \pi$ and $\omega t = 10$. In the limit $\omega t \to \infty$, $e(\eta, \omega t)$ (18) becomes real (lower plot).

$T/(\partial U/\partial j)|_{j_c}$, where $U(j)$ is an activation energy and j_c is the critical current density measured at $E = E_c$ such that $U(j_c) = 0$. In particular, for $U(j) = U_0 \ln(j_c/j)$, one has $E(j) = E_c(j/j_c)^n$ with $n = U_0/T \gg 1$ and $j_\rho \approx j_c/n$ [16–18]. The current density $j(\eta, t)$ and magnetic moment $M(t)$ are then obtained by inserting the universal $E_0(\eta, t)$ (1) into the specific $E(j)$ law.

In this Letter we calculate the linear response of thin superconductor strips and disks in the flux-creep regime to a small transverse ac magnetic field, by combining the flux-creep theory [15] with the dynamic linear response theory of thin Ohmic conductors [10]. The problem is that in the flux-creep state the background electric field and thus the differential resistivity $\rho(E)$ decay with time as $1/t$, so not only the calculation but also the notion of the proper linear response becomes nontrivial. Nevertheless, this calculation can be performed exactly even for the common transverse geometry with strong demagnetization effects, where it amounts to the solution of a nonlinear and *nonlocal* diffusion equation. A striking feature of this case is that the obtained linear ac susceptibility $\chi(\omega)$ is *independent of any material parameter* if the nonlinearity of $E(j)$ is sufficiently strong, that is, $s = d\ln j/d\ln t \ll 1$. This condition always holds well below the irreversibility field. A similar universal ac response was predicted for longitudinal geometry [19].

The expression for $\chi(\omega)$ that we derive for the nonlinear flux-creep state is similar to the $\chi(\omega)$ of linear conductors of the same geometry [10],

$$\chi(\omega) = -\frac{4}{\beta}\sum_n \frac{d_n^2 \Lambda_n}{1 + \Lambda_n/(i\omega\tau)}. \qquad (3)$$

However, unlike the well-defined Ohmic relaxation time $\tau = ad\mu_0/2\pi\rho$ in (3), which in general may be complex if the linear resistivity $\rho = \rho_{ac}(\omega)$ is complex, the flux-creep "time constant" itself depends on t via $\rho(E)$. Using Eqs. (1) and (2) for a qualitative estimate of τ, one can show that in superconductors τ should be just replaced by the time t elapsed after the creep has started. In both theories the constants Λ_n and d_n are eigenvalues and oscillator strengths of eigenvalue problems which differ by their integral kernels, and $\beta = 1$ for strips and $\beta = 32/3\pi^2 = 1.08$ for disks. For $\omega\tau \to \infty$ the sum (3) yields the ideal diamagnetic response $\chi = -1$.

We start our calculation from the basic integral equation which describes a nonlinear nonlocal diffusion of $E(\eta, t)$ for perpendicular geometry [8,10,15],

$$E(\eta, t) = -\gamma \dot{B}_a(t)\eta + \alpha \int_0^1 \frac{\dot{E}(u,t)}{E(u,t)} Q_0(\eta, u)\, du. \qquad (4)$$

Here $\alpha = ad\mu_0 j_\rho/2\pi$, $\gamma = 1$ ($\gamma = \frac{1}{2}$) for strips (disks), and the integral kernel Q_0 equals

$$Q_0(\eta, u) = \ln\left|\frac{\eta - u}{\eta + u}\right|, \qquad (5)$$

$$Q_0(\eta, u) = \left(\frac{u}{\eta}\right)^{1/2} k \int_0^{\pi/2} \frac{1 - 2\sin^2\phi}{(1 - k^2\sin^2\phi)^{1/2}}\, d\phi \qquad (6)$$

for strips and disks, respectively, with $k^2 = 4\eta u/(\eta + u)^2$ [8,10]. For zero ramp rate $\dot{B}_a = 0$, the steady-state creep solution of (4) is given by Eq. (1) with [15]

$$f(\eta) = -\int_0^1 Q_0(\eta, u)\, du. \qquad (7)$$

Applying a small ac field $B_1(t)$ on top of the constant dc magnetic field, one obtains an equation for the induced electric field $E_1(\eta, t)$ by replacing E in (4) by $E_0 + E_1$. We thus get the integral equation for $E_1(\eta, t)$,

$$E_1(\eta, t) = -h(t)\eta + \int_0^1 Q(\eta, u)[t\dot{E}_1(u,t) + E_1(u,t)]\, du \qquad (8)$$

with $Q(\eta, u) = Q_0(\eta, u)/f(u)$ and $h(t) = \gamma a \dot{B}_1(t)$. The solution of Eq. (8) can be written in the form

$$E_1(\eta) = g(\eta)\sum_n c_n(t)\varphi_n(\eta), \qquad (9)$$

where $\varphi_n(\eta)$ are the orthogonal normalized eigenfunctions which describe dissipative flux-creep modes $E_{1,n}(\eta, t) \propto \varphi_n(\eta)t^{-\Lambda_n}$. The $\varphi_n(\eta)$ and Λ_n are determined by the following eigenvalue problem,

$$\varphi_n(\eta) = -\Lambda_n \int_0^1 \tilde{Q}(\eta, u)\varphi_n(u)\, du, \qquad (10)$$

$$\int_0^1 \varphi_n(u)\varphi_m(u)\, du = \delta_{mn}. \qquad (11)$$

Here $\tilde{Q}(\eta, u) = \tilde{Q}(u, \eta) = Q(\eta, u)g(u)/g(\eta)$ is a symmetric kernel, where $g(u) = f(u)^{1/2}$ for strips and $g(u) = [f(u)/u]^{1/2}$ for disks, $f(u)$ is given by (7), and δ_{mn} is the Kronecker symbol.

We have solved Eq. (10) numerically on a grid of up to 300 nonequidistant points η_i as described in [8,10]. The first eigenvalues are $\Lambda_1, \ldots, \Lambda_5 = 1$, 1.9029, 2.6673, 3.3589, 4.0005 (1, 1.8156, 2.4885, 3.0932, 3.6561) for the strip (disk), and $\Lambda_n \approx 0.6n$ for $n \gg 1$. The first eigenfunction is $\varphi_1(u) = f_s(u)^{1/2}$ for strips and $\varphi_1(u) = [uf_d(u)]^{1/2}$ for disks. The eigenfunctions $\varphi_n(u)$ exhibit logarithmically diverging slopes at $u = 0$ and $u = 1$ and very narrow oscillations near $u = 0$.

The equation for $c_n(t)$ can be obtained by multiplying Eq. (8) by $\varphi_n(\eta)$ and integrating over η with the use of (10) and (11). This yields

$$t\dot{c}_n + (1 + \Lambda_n)c_n = d_n h(t), \qquad (12)$$

$$d_n = \int_0^1 \frac{\varphi_n(\eta)\eta}{g(\eta)}\, d\eta, \qquad (13)$$

where the first "oscillator strengths" d_n are $d_1, \ldots, d_5 = 0.4247$, 0.1291, 0.0651, 0.0409, 0.0288 (0.4476, 0.1308. 0.0661, 0.0425, 0.0310) for strips (disks) and $d_n \approx 0.36/n^{3/2}$ for $n \gg 1$.

Substituting the solution of Eq. (12) into Eq. (9) we obtain a general solution of the linear response problem,

$$E_1(\eta, t) = g(\eta) \sum_n \frac{d_n \Lambda_n \varphi_n(\eta)}{t^{1+\Lambda_n}} \int_{t_0}^{t} h(t_1) t_1^{\Lambda_n} \, dt_1 \,. \quad (14)$$

Here we assume that $h(t)$ was switched on at $t = t_0$, later than the transient time for flux creep [14,15].

Now we consider the important particular case of a periodic ac signal $B_1(t) = B_0 \exp(i\omega t)$ in more detail. For $\omega t \gg 1$ an asymptotic solution of Eq. (12) is

$$c_n(t) = i\omega\gamma a d_n B_0 e^{i\omega t} \times \left[\frac{1}{\Lambda_n + i\omega t} - \frac{\Lambda_n}{(\Lambda_n + i\omega t)^3} + \ldots \right]. \quad (15)$$

Substituting Eq. (15) into Eq. (9) we obtain

$$E_1(\eta, t) = \frac{a B_0 \exp(i\omega t)}{t} e(\eta, \omega t) \,, \quad (16)$$

$$e(\eta, \omega t) = \gamma g(\eta) \sum_n \frac{d_n \Lambda_n \varphi_n(\eta)}{1 + \Lambda_n/(i\omega t)} \,. \quad (17)$$

Here only the first term in the square brackets in Eq. (15) was retained since the remaining terms give contributions of higher order in $1/\omega t \ll 1$.

For $\omega t \to \infty$ the function $e(\eta, \omega t)$ in (17) equals

$$e(\eta, \infty) = (\gamma'/\pi) f(\eta) \eta/(1 - \eta^2)^{1/2} \quad (18)$$

with $\gamma' = 1$ ($\gamma' = 2/\pi$) for strips (disks) and $f(\eta)$ from (7). This means that the electric field (16) for $\omega t \to \infty$ is caused by the additional current density

$$j_1(\eta, t) = j_\rho E_1/E_0 = 2\gamma' B_1(t) \eta/(1 - \eta^2)^{1/2}, \quad (19)$$

which is just the current that ideally screens the applied ac field from the interior of the strip or disk [8]. The ac screening current (19) is superimposed to the background time-decaying current $j_0 \approx j_c$ obtained by inverting the relation $E(j_0) = E_0(\eta, t)$.

As seen from Fig. 1, at finite $\omega t < \infty$ the perturbation $E_1(\eta)$ is rounded and finite at the edges, and gains a dissipative part since $e(\eta, \omega t)$ (17) becomes complex, similar to the $j(r, \omega)$ profiles of the Ohmic disk in Ref. [8]. The ac component $j_1 = E_1(\partial E/\partial j)^{-1} = E_1 j_\rho/E_0$ perturbs the magnetic moment $M = \frac{1}{2}\hat{\mathbf{z}} \int \mathbf{j} \times \mathbf{r} \, d^3r = M_0(t) + M_1(t)$, yielding for a disk $M = \pi d \int_0^a r^2 j(r) \, dr$ and

$$M_1(t) = \pi d a^3 j_\rho \int_0^1 \eta^2 \frac{E_1(\eta, t)}{E_0(\eta, t)} \, d\eta \,. \quad (20)$$

Note that in j_1 and M_1 (17) the factors $1/t$ from E_0 (1) and E_1 (16) cancel. The amplitudes of j_1 and of $M_1(t) = M_1(\omega) \exp(i\omega t)$ remain thus nearly *constant* during creep, although E_0, E_1, and j_0 decrease with t.

From (1), (16), and (20) the ac susceptibility $\chi(\omega) = \mu(\omega) - 1 = -M_1(\omega)/M_1(\omega = \infty)$ for $\omega t \gg 1$ is obtained,

$$\chi(\omega) = -\frac{4}{\beta} \sum_n \frac{d_n^2 \Lambda_n}{1 + \Lambda_n/i\omega t} \,, \quad (21)$$

where $\beta = 1$ (strips), $\beta_d = 32/(3\pi^2) = 1.08$ (disks), and the Λ_n and d_n are determined by Eqs. (10) and (13). Evaluating (21) numerically, we find to a very good accuracy [relative error <0.001 (0.02) for $\omega t \geq 30$ (10)]

$$\chi(\omega) = -1 + p \frac{\ln(q i \omega t)}{i\omega t} \quad (22)$$

with $p = 0.2804$ (0.2524) and $q = 19.85$ (28.74) for strips (disks). The real and imaginary parts of $\chi = \chi' - i\chi''$ (22) take the forms

$$\chi'(\omega) = -1 + p\pi/2\omega t \,, \quad (23)$$

$$\chi''(\omega) = p \ln(q\omega t)/\omega t \,. \quad (24)$$

This linear ac susceptibility depends only on the creep time t and sample shape, but it is independent of any material parameter and of T and B. Such a universality was also noted for longitudinal geometry [19]. Therefore, in longitudinal geometry at $\omega t \gg 1$ in general

$$\chi(\omega) = -1 + (1 - i) \, \mathrm{const}/\sqrt{\omega t} \,, \quad (25)$$

where the constant depends only on the shape of the specimen cross section. The linear $\chi(\omega)$ (21)–(25) should not be confused with the nonlinear hysteretic ac response caused at low dc fields, e.g., by surface barriers [20].

For comparison we note that the ac susceptibility of Ohmic conductors (3) is given by the same expressions (21)–(24) but with ωt replaced by $\omega\tau \gg 1$ and $p = 2/\pi^2 = 0.2026$ ($p = 3/\pi^2 = 0.3040$), $q = 16.2$ ($q = 11.3$) for strips (disks) [10]. These constants are of the same order as for the above nonlinear conductor. The general formula (3) for the Ohmic $\chi(\omega)$ applies for all ω; its $\chi''(\omega)$ has a maximum at $\omega\tau = 1.108$ (1.169) for the strip (disk). In contrast, our $\chi(\omega)$ for the creep regime makes sense only if $\omega t \gg 1$, since at least a few ac cycles should be completed during the creep time t.

Notice that the creep susceptibility (21) to (24) becomes independent of time if $\omega = \omega(t) \approx \Omega/t$ is chosen. This corresponds to a constant frequency Ω on a logarithmic time scale since $B_1(t) = B_1 \cos(\Omega \ln t)$ yields $\omega = \partial(\Omega \ln t)/\partial t = \Omega/t$. For a disk this gives the constant-amplitude response

$$M_1(t) = (32/3\pi^2\mu_0) B_1 a^3 \chi' \cos(\Omega \ln t - \theta) \quad (26)$$

with the phase shift $\theta = \arctan(\chi''/\chi') > 0$. We have checked these predictions by direct computation of perturbed flux creep. The numerical method is described in Refs. [8,15]. We use a power law $E(j) = E_c |j/j_c|^n \, \mathrm{sgn} j$ [16–18], which means $j_\rho \approx j_c/n \ll j_c$. These computations confirm our analytical results for E_0, E_1, j_0, j_1, M_0, and M_1 as functions of η and t; see Figs. 1 and 2.

Remarkably, while the linear responses E_1 and j_1 strictly speaking are valid only for small ac amplitudes

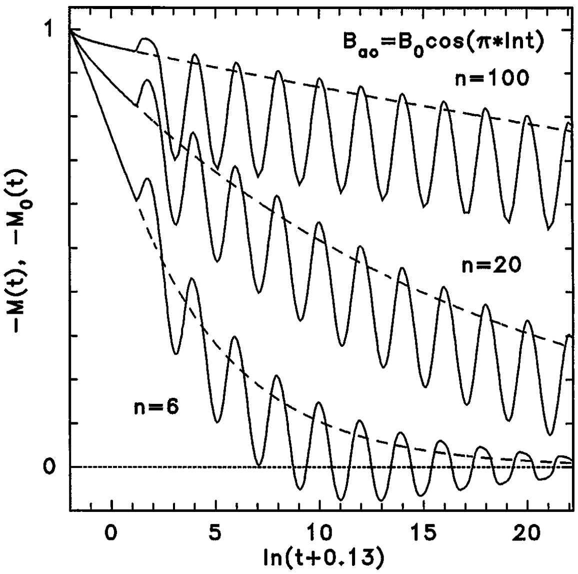

FIG. 2. The computed magnetic moment $M(t)$ during flux creep perturbed by an ac field $B_1(t) = 0.05\mu_0 j_c d \cos(\Omega \ln t)$ with $\Omega = \pi$, plotted versus $\ln(t + 0.13)$ ($a = j_c d = E_c = 1$) and normalized to $M(0) = 1$ for three creep exponents $n = 6$, 20, and 200 of the current-voltage law $E \propto j^n$. The dashed lines show the unperturbed $M_0(t)$. Because of the large ac amplitude, the depicted nonlinear ac response $M_1 = M - M_0$ is shifted downward and is larger than the linear response from Eq. (23) (which is good even for our small $\omega t = \pi$) but still smaller than the ideal Meissner response $\chi = -1$. The phase shift θ also falls between these two limits, $0 < \theta < \arctan(\chi''/\chi')$.

$B_1 \ll B_{\text{lin}} = \mu_0 j_\rho d$ for which $E_1 \ll E_0$, the linear Meissner-like response in $M_1(t)$ was observed in our computer simulations up to much larger ac amplitudes of order the penetration field $B_p \approx \mu_0 j_c d/\pi \approx B_{\text{lin}} n/\pi$. Moreover, the ac component $M_1(t)$ looks smoothly sinusoidal (rather than being clipped) even at very large ac amplitudes where $|M_1(t)|$ exceeds the background $M_0(t)$ and thus the Bean critical state should be reversed periodically. The presence of flux creep thus smooths the quasistatic periodic response predicted by the Bean model. In particular, for $B_{\text{lin}} < B_1 < B_p$, j does not fully invert to $-j_c$, and the electric field $E(j)$ thus stays practically zero near the specimen edges most of the time [21].

In conclusion, we have calculated the transverse linear ac response of a superconductor strip or disk in the flux-creep regime, equivalent to a highly nonlinear conductor. The resulting ac susceptibility is similar to that for linear conductors with the relaxation time replaced by the creep time t, and thus is *independent* of any material parameter. These results are born out by direct computations of flux creep in the presence of a small ac field that allow one to also probe the onset of nonlinear ac response. This nonlinear response will be dealt with elsewhere [21].

The work of A. G. was supported by NSF MRG Program (DRM-9214707).

[1] P. N. Mikheenko and Yu. E. Kuzovlev, Physica (Amsterdam) **204C**, 229 (1993).
[2] E. H. Brandt, M. V. Indenbom, and A. Forkl, Europhys. Lett. **22**, 735 (1993); E. H. Brandt and M. V. Indenbom, Phys. Rev. B **48**, 12 893 (1993).
[3] E. Zeldov *et al.*, Phys. Rev. B **49**, 9802 (1994); Phys. Rev. Lett. **73**, 1428 (1994); M. McElfresh *et al.*, Phys. Rev. B **51**, 9111 (1995).
[4] E. H. Brandt, Phys. Rev. Lett. **74**, 3025 (1995); W. Xing *et al.*, J. Appl. Phys. **76**, 4244 (1994); Th. Schuster *et al.*, Phys. Rev. B **52**, 10 375 (1995); E. H. Brandt, Phys. Rev. B **52**, 15 442 (1995); Rep. Prog. Phys. **58**, 1465 (1995).
[5] Th. Schuster *et al.*, Phys. Rev. Lett. **73**, 1424 (1994); P. D. Grant *et al.*, Physica (Amsterdam) **229C**, 289 (1994); Th. Schuster *et al.*, Phys. Rev. B **50**, 16 684 (1994); R. Wijngaarden *et al.* (unpublished).
[6] J. R. Clem, H. R. Kerchner, and T. S. Sekula, Phys. Rev. B **14**, 1893 (1976).
[7] P. H. Kes, J. Aarts, J. van der Berg, C. J. van der Beek, and J. A. Mydosh, Supercond. Sci. Technol. **1**, 242 (1989).
[8] E. H. Brandt, Phys. Rev. Lett. **71**, 2821 (1993); Phys. Rev. B **49**, 9024 (1994); Phys. Rev. B **50**, 4034 (1994).
[9] M. A. Skvortsov and V. B. Geshkenbein, Zh. Eksp. Teor. Fiz. **105**, 1379 (1994) [Sov. Phys. JETP **78**, 743 (1994)].
[10] E. H. Brandt, Phys. Rev. B **50**, 13 833 (1994).
[11] J. Kötzler, G. Nakielski, M. Baumann, R. Behr, F. Goerke, and E. H. Brandt, Phys. Rev. B **50**, 3384 (1994).
[12] A. T. Dorsey, Phys. Rev. B **51**, 15 329 (1995).
[13] J. Gilchrist, Physica (Amsterdam) **219C**, 67 (1994).
[14] A. Gurevich and H. Küpfer, Phys. Rev. B **48**, 6477 (1993).
[15] A. Gurevich and E. H. Brandt, Phys. Rev. Lett. **73**, 178 (1994); A. Gurevich, Int. J. Mod. Phys. B **9**, 1045 (1995).
[16] V. M. Vinokur, M. V. Feigel'man, and V. B. Geshkenbein, Phys. Rev. Lett. **67**, 915 (1991).
[17] J. Rhyner, Physica (Amsterdam) **212C**, 292 (1993).
[18] J. Gilchrist and M. Konczykowski, Physica (Amsterdam) **212C**, 43 (1993).
[19] V. B. Geshkenbein, M. V. Feigel'man, and V. M. Vinokur, Physica (Amsterdam) **185–189C**, 2511 (1991).
[20] N. Morozov, E. Zeldov, D. Majer, and B. Khaykovich, Phys. Rev. Lett. **76**, 138 (1996).
[21] A. Gurevich and E. H. Brandt (unpublished).