New Scenario for Transition to Turbulence?

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Numerical study of the one-dimensional nonlinear partial differential equation, equivalent to that proposed [*Recent Advances in Engineering Science* (Springer-Verlag, Berlin, 1989)] to describe longitudinal seismic waves, is presented. The equation has a threshold of short-wave instability and symmetry, providing slow long-wave dynamics. It is shown that the threshold of the short-wave instability corresponds to a point of "continuous" (second order) transition from a spatially uniform state to a chaotic regime. Thus, contrary to the conventional scenarios, turbulence arises from the spatially uniform state as a result of *one and the only one* supercritical bifurcation.

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In some cases a pattern-forming extended dissipative system has, besides the trivial symmetry under a shift of the origin of the spatiotemporal coordinate frame, an additional group of continuous symmetry. Such examples are convection in a liquid layer with stress-free boundary conditions [1,2], systems with Galilean invariance [3], a traveling front in phase transition phenomena or in reaction-diffusion systems [4,5], electroconvection in liquid crystals with a homeotropic alignment of the director beyond the threshold of the Frederiksz transition [6], and others. It is well known that the additional symmetry generates an extra band of slowly varying modes, i.e., adds extra degrees of freedom to the order parameter describing the dynamics of pattern formation [7], that brings about dramatic changes in the *pattern stability* problem. Among other things, the most important to the present paper is instability of all weakly nonlinear spatially periodic patterns that may occur in such systems [2,8]. In particular, it takes place [8,9] with solutions of the equation

$$\frac{\partial u}{\partial t} + \frac{\partial^2}{\partial x^2}\left[\varepsilon - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2\right]u + \left(\frac{\partial u}{\partial x}\right)^2 = 0, \quad (1)$$

whose additional symmetry transformation is $u \to u +$ const. Here the quantities $u(x,t)$ and ε are real and the control parameter ε supposed to be small. At $\varepsilon > 0$ the trivial solution of Eq. (1) $u \equiv 0$ undergoes short-wave instability against spatially periodic perturbations with wave numbers k from a narrow band centered around $k = 1$.

An equivalent form of Eq. (1) was introduced in Ref. [10] to govern longitudinal seismic waves in viscoelastic media and later was analyzed in Ref. [11], where a finite range of stability of steady spatially periodic patterns was obtained. However, this result of Ref. [11] is wrong due to inadequate truncation of amplitude equations. For more details, see Ref. [9].

Note that the similarity of the pattern stability problem in different systems with the same dimensionality of the continuous group of symmetry (the number of continuous quantities parametrizing the whole set of invariant transformations) [8] provides us with grounds to expect that qualitative features of Eq. (1) are not specific to this particular equation but common to many different systems with additional symmetry too.

It is well known that close to the threshold of short-wave instability a pattern-forming system has a certain hierarchy of scales, so that the solution of the governing equation(s) may be presented as a product of a steady, rapidly varying in space periodic function, and a slowly varying, both in space and in time, amplitude (envelop) [7]. The important property of systems with additional symmetry is that, contrary to the conventional cases without this symmetry, the control parameter cannot be scaled out from equations for the slowly varying amplitudes. There are at least two reasons for that. Firstly, at $0 < \varepsilon \ll 1$ the relevant problem has *three* bands of slowly varying in time modes, centered around $k = \pm 1$, 0, respectively, and the equations for envelops to modes with $k = \pm 1$ have the structure different from that to modes with k close to zero [11]. Secondly, the problem is characterized by mixing of different scales in perturbative expansion [2,5,8] that forced one to take into account corrections to a leading approximation to the amplitude equations. Thus, it is trivial to see that the number of degrees of freedom connected with rescaling of independent variables is not enough to scale out ε entirely. In such a case interplay of different scales may result in a complicated pattern dynamics at any finite positive ε, no matter how small it is.

Instability of all steady spatially periodic solutions to Eq. (1) just beyond the threshold of the short-wave instability of the trivial state raises a question about the state the unstable spatially uniform one evolves to. To answer the question we employed numerical study of Eq. (1). The simulations were developed in a finite segment $0 \le x \le L$ with periodic boundary conditions. To avoid difficulties of approximation by finite differences of the high-order differential operator of Eq. (1) we used the Fourier transform of $u(x,t)$

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with respect to the spatial variable. As a result Eq. (1) was reduced to a set of coupled ordinary differential equations of the first order for amplitudes $U_m(t)$ of Fourier modes with wave numbers $k = mp$, where m is an integer and $p \equiv 2\pi/L$. Since the mode $U_0(t)$ is slaved to those with $m \neq 0$ [9] and plays no role in the pattern-formation problem, this mode was not taken into account. To understand the behavior of the system just beyond the threshold of the short-wave instability and to avoid complications associated with secondary bifurcations, extremely small values of ε were chosen. The simulations were carried out on a CRAY C-90 supercomputer. A detailed description of the code will be reported elsewhere. Here we mention only that the code was written especially to deal with small ε and was carefully tested against all available analytical results.

The numerical study shows that any small-amplitude initial spatial distribution of the order parameter $u(x,0)$ evolves, after a certain transient period, into one or another time-dependent asymptotic regime. All these regimes are characterized by excitation of bunches of modes from narrow subbands centered around the points $k = \pm n$, where the integer n may be regarded as a number of the corresponding subband. Amplitudes of the modes fall off rapidly to zero with increase in both the deviation of k from the center of each subband, at fixed n, and the number n, at the fixed deviation.

To classify the asymptotic regimes note that, since long-wave dynamics is very important for the problem under consideration, the regimes may be very sensitive to the cutoff of long-wave modes caused by finiteness of L (size effect), i.e., to a particular value of p. In order to study this effect the following approach was developed. A run with a certain fixed value of p lasted until the asymptotic state was reached. Then, the simulation terminated, p replaced by $p/2$, the state obtained at the moment of termination, perturbed by a small-amplitude mode with the wave number $k = p/2$, used as the initial condition for a new run, and so on. The study shows that if p is not small enough, so that only a few Fourier modes fall into the band of instability of the trivial state $u \equiv 0$, the asymptotic regime corresponds to nonlinear periodic oscillations of U_m (a limit cycle). A typical example is presented in Fig. 1. Each decrease of p yields instability of the limit cycle and its transformation either into a more complicated one or (finally) into a chaotic state; see Fig. 2. An example of a well-developed chaotic state is shown in Figs. 3 and 4. In the phase space of the amplitudes U_m the regime, corresponding to these figures, is characterized by exponential growth of the separation of two trajectories starting out close to each other. The study of the power spectra and correlation functions for modes from different subbands indicates that each subband has a certain characteristic time τ_n that is nothing but the characteristic time scale of the fine structure of the dependence $U_m(t)$; see Figs. 3(b) and 4(b). Note that

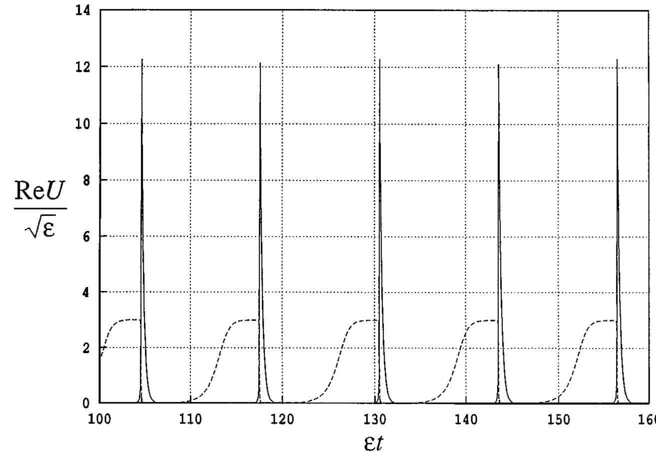

FIG. 1. A limit cycle obtained as an asymptotic state at $\varepsilon = 10^{-4}$, $p = 0.02$. The time dependence of the real part of the amplitudes of the modes with $k = p$ (—) and $k = 1$ (− −) is shown as an example.

τ_0 is considerably larger than τ_1; cf. Figs. 3(b) and 4(b). We have no reliable data to conclude if the characteristics of well-developed chaotic regimes undergo any change with further decrease of p.

Extending the results to the case of the boundless space ($L \rightarrow \infty$), we arrive at the following conclusion.

(1) Instability, arising in the problem at *any* small positive ε, drives the systems into well-developed spatiotemporal chaotic states.

(2) The effective phase spaces of these states have high (continuous?) dimensionality and contain a wide variety of limit cycles that are stable in some directions and unstable in the others.

(3) Chaos corresponds to random "scattering" of phase trajectories by the limit cycles (attraction along stable directions with repulsion along unstable ones).

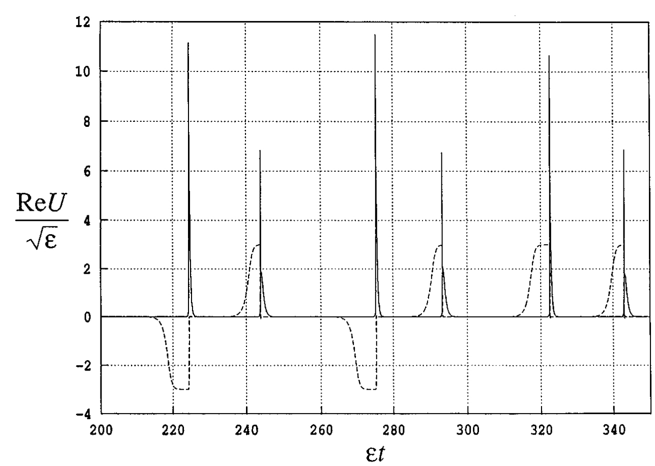

FIG. 2. The same quantities as those shown in Fig. 1. The asymptotic state at $p = 0.01$.

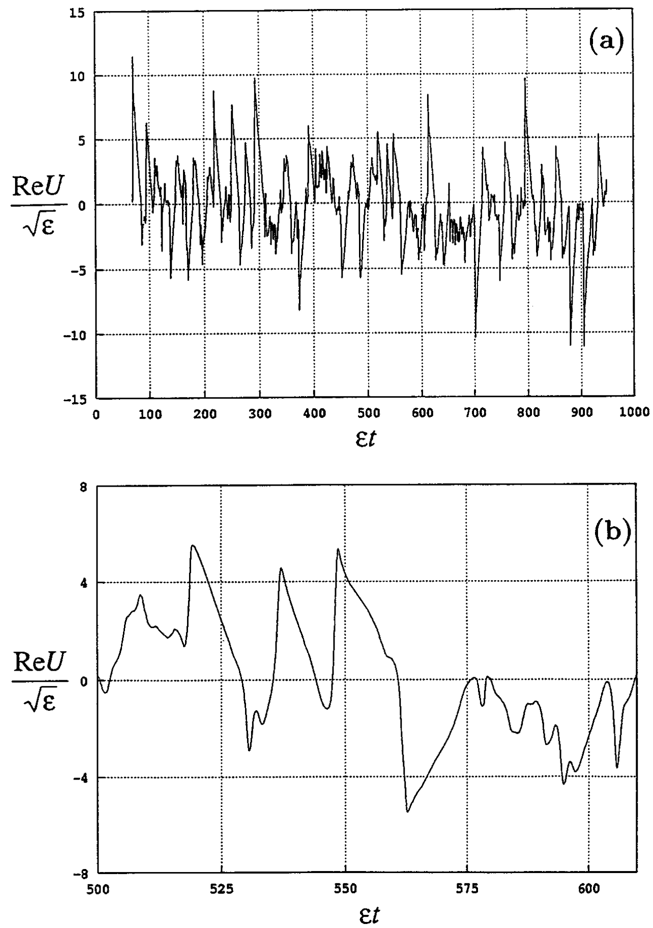

FIG. 3. Well-developed chaotic regime, $\varepsilon = 10^{-4}$, $p = 3.125 \times 10^{-3}$. (a) Real part of the amplitude of the mode with $k = p$ versus time. (b) A small fragment of the same curve, clearly indicating presence of the characteristic time in the fine structure of the curve.

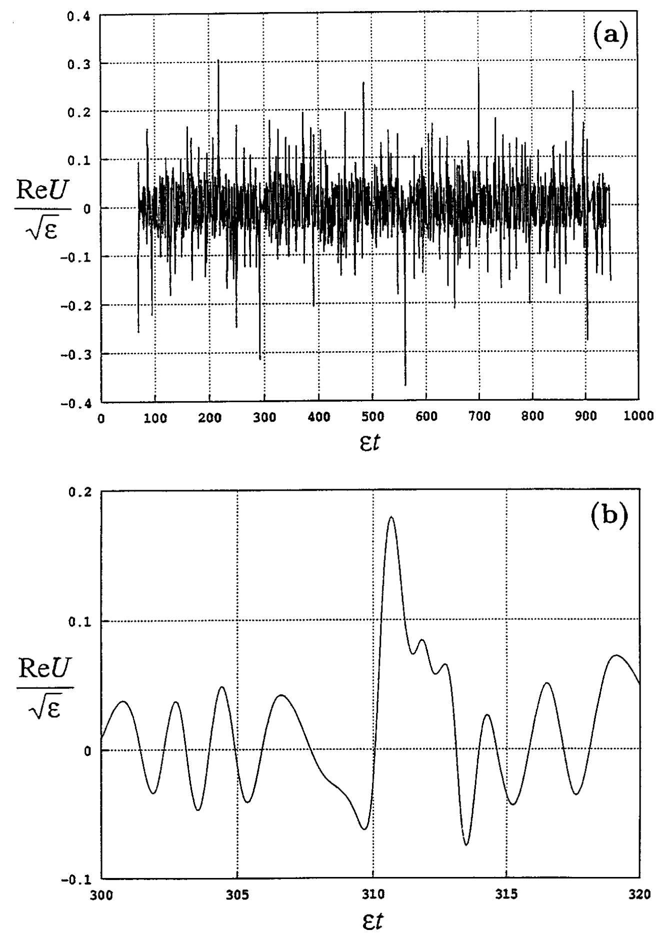

FIG. 4. The same as that shown in Fig. 3 for the mode with $k = 1$.

(4) Chaotic dynamics of different modes have a fine structure with a certain characteristic time scaled as $1/\varepsilon$ approximately. The characteristic time diverges at $\varepsilon \to 0$, so that at small ε such a dynamics may be called *slow turbulence*.

Note that contrary to the conventional equations, exhibiting a chaotic behavior, such as, e.g., the Ginzburg-Landau equation with complex coefficients, in our case the small bifurcation parameter cannot be eliminated from the problem, as already emphasized above, so the slow turbulence is really *slow*.

The amplitudes of chaotic modes at slow turbulence are scaled approximately as $\sqrt{\varepsilon}$ that indicates the *supercritical* (normal) bifurcation. However, it is difficult to provide high accuracy for the scaling, treating data related to chaotic dynamics. Therefore it is desirable to obtain more evidence of the supercritical character of the bifurcation. Such evidence was obtained in the simulations by "adiabatic" scanning of ε from zero to a small positive value and backward that did not indicate any hysteretic phenomena.

It may seem that the extremely small values of the bifurcation parameter considered in the present paper are meaningless since the governing equation does not include fluctuations. On the other hand, it is known that in equilibrium phase transition phenomena close to the transition point fluctuations can change the type of bifurcation from *supercritical* to weakly *subcritical* [12]. However, contrary to this case, in *macroscopic* pattern-forming systems the relevant dimensionless parameter, characterizing the effect, contains the ratio of an atomic correlation length to a macroscopic pattern's scale [13] and the above effect always is negligible [14].

Thus, the considered dissipative system with short-wave instability and additional symmetry, generating slow long-wave dynamics, does exhibit a new scenario of transition to turbulence which is equivalent to second order phase transitions in equilibrium systems.

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