Wigner Random Banded Matrices with Sparse Structure: Local Spectral Density of States

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Random banded matrices with linearly increasing diagonal elements are recently considered as an attractive model for complex nuclei and atoms. Apart from early papers by Wigner there were no analytical studies on the subject. In this Letter we present analytical and numerical results for local spectral density of states (LDOS) for a more general case of matrices with a sparsity inside the band. The crossover from the semicircle form of LDOS to that given by the Breit-Wigner formula is studied in detail.

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Recently, there was a growing interest in the statistical properties of random banded matrices (RBM) (see, e.g., [1], and references therein). These $N \times N$ matrices H_{nm} can be characterized as those containing nonzero elements only within a wide band of the size $b \ll N$ around the main diagonal. It was realized that matrices with such a structure are appreciably different in their statistical properties from those forming the classical Gaussian ensembles studied by many authors [2]. More precisely, the RBM of infinite size do not show the famous effect of level repulsion that is the direct consequence of the localized nature of their eigenvectors [3–5].

The most studied type of RBM is that with the zero mean value of all matrix elements and with the variance given by $\overline{H_{nm}^2} = V^2 a(|n - m|/b)$, the function $a(r)$ decaying exponentially (or faster) at $r \gg 1$. Such matrices were shown to be relevant for understanding some properties of the quantum kicked rotator, one of the paradigmatic models in the domain of quantum chaos [4]. On the other hand, the matrix H_{nm} can be treated both as a tight-binding Hamiltonian of a quantum particle in a 1D system with long range random hoppings and as an adequate model for quasi-1D disordered wires [6]. In a number of publications properties of such matrices were studied both numerically and analytically; see references in [1]. In particular, it was found that for matrices of infinite size any eigenvector has a typically finite number (of the order of $l_\infty \propto b^2$) of essentially nonzero components. When the matrix is of a large finite size $N \gg b \gg 1$, its statistical properties were shown to be determined by the scaling parameter b^2/N [5,6]. As to the density of states (DOS) it has generally a form of the semicircular law [7]

$$\rho(E) = \begin{cases} \frac{2}{\pi R_0^2}\sqrt{R_0^2 - E^2}, & |E| \le R_0, \\ 0, & |E| > R_0, \end{cases} \quad (1)$$

with $R_0 \propto b^{1/2}$ being the semicircle radius.

Another type of the RBM—that with the mean value of diagonal elements increasing linearly along the main diagonal: $\overline{H_{nn}} = \alpha n$—has attracted recently a lot of research activity [8–13]. In a tight-binding analogy mentioned above these matrices describe a quantum particle in a 1D disordered system subject to a constant electric field. Another interpretation was developed in [9,11] where matrices of such kind are considered as Hamiltonians of generic conservative systems with complex behavior, like heavy atoms and nuclei. Indeed, a very detailed study of compound states in the chaotic Ce atom [13,14] has revealed that the latter class of the RBM can be used rather efficiently to describe such a physical system. This particular kind of RBM was introduced for the first time by Wigner about 40 years ago [15]. For this reason we refer to this ensemble as that formed by Wigner random banded matrices (WRBM). Quite close ideas were developed recently [12] where the model for a general integrable Hamiltonian H_0 perturbed by some generic perturbation V_0 was investigated. It was argued that the matrix $(H_0 + V_0)_{nm}$ in the basis of semiclassical eigenstates of the unperturbed part H_0 has the form of *sparse* WRBM.

As is known, much more informative characteristics compared to the global DOS is the so-called *local spectral density of states* (in the literature, LDOS). The latter function, also known in nuclear physics as *strength function,* is defined as

$$\rho(E, n) = \sum_\nu |\psi_\nu(n)|^2 \delta(E - E_\nu), \quad (2)$$

where $\psi_\nu(n)$ is the nth component of the eigenvector ψ_ν corresponding to the energy level E_ν, $n = 1, 2, \ldots, N$. After averaging over the disorder the LDOS depends (in the limit $N \longrightarrow \infty$) on the parameter $z = E - \alpha n$ only: $\overline{\rho(E, n)} = \rho_L(z)$. As a result, the global DOS is energy independent: $\overline{\rho(E)} = \frac{1}{N}\sum_{n=1}^N \rho(E, n) \to \frac{1}{\alpha N}$ provided the parameter α is small enough for the sum being replaced by the integral [16].

The quantity of the most physical interest is the width Γ of the LDOS: It generates a new length scale $l_{max} = \Gamma/\alpha$; see details in [6,9,11]. If one treats linearly increasing diagonal elements as eigenvalues of an unperturbed Hamiltonian, the off-diagonal elements play a role of perturbation and the length l_{max} can be interpreted as an effective number of unperturbed states coupled by the perturbation [9]. In the limit $l_{max} \ll 1$ standard perturbation theory can be applied since the perturbation is weak. We will refer to this regime as the *perturbative* one. In the opposite limit $l_{max} \gg 1$ the problem is essentially nonperturbative. The latter case is our main concern in the present paper.

In the case of very large α well inside the perturbative regime the form of LDOS is determined mostly by the distribution of diagonal elements (e.g., Gaussian). On the other hand, for α in the nonperturbative regime the region of parameters was indicated in [17] where the Lorentzian shape

$$\rho_{BW}(E, n) = \frac{\Gamma/2\pi}{(E - \alpha n)^2 + \Gamma^2/4} \qquad (3)$$

is expected with $\Gamma \approx 2\pi\rho V^2$ [18]. Actually, this form of LDOS was obtained by Wigner in his early studies of WRBM [15] as a result of some limiting procedure.

Equation (3) is a commonly accepted approximation for the LDOS in nuclear physics known as the Breit-Wigner (BW) formula. It is considered to be well in agreement with experimental data for nuclei [19] and complex atoms [13]. This fact makes the WRBM much more attractive as a model for physical systems as compared with the full random matrices. However, a broader application of this ensemble is restricted by the absence of a general theoretical understanding of their properties. Indeed, the method used in Wigner's pioneering papers [15] is rather involved, partially heuristic, and seems to be based upon a specific form of the parameters chosen. In particular, it is quite unclear how universal are Wigner results [e.g., the Breit-Wigner form of the LDOS, Eq. (3)] against variations of the parameters α, b, form of the distribution of the off-diagonal elements, and, ultimately, the sparsity [20].

Motivated by all this, we have performed an analytical consideration of the LDOS for general WRBM with an arbitrary degree of sparsity. For this purpose we consider random symmetric matrices $H_{nm} = \alpha n \delta_{nm} + t_{nm}$, where the probability distribution of $t_{nm}, n \leq m$ is given by

$$\mathcal{P}(t_{nm}) = (1 - p_{nm})\delta(t_{nm}) + p_{nm}h_1(t_{nm}), \qquad (4)$$

with $h_1(t) = \frac{1}{V}h(t/V)$ being a generic distribution with zero mean and the variance V^2. The probability p_{nm} of being nonzero for any element is taken to be of the form $p_{nm} = \frac{M}{b}f(|n - m|/b)$, where $f(t)$ decays exponentially or faster at infinity and $\sum_{r=0}^{\infty}\frac{1}{b}f(r/b) = 1$. In such a definition the mean number of nonzero elements

per row is just $2M$ and the quantity $S = b/M$ is natural to term "sparsity."

In order to calculate the LDOS let us follow the method used previously for sparse matrices without band structure [21]. Relegating all technical details to a more extended publication, let us just mention that the mean local DOS is expressed in the form of a functional integral, which is finally calculated by the saddle-point approximation exploiting the parameters $b \gg 1$ and $N \gg 1$. Unfortunately, it is hard to get a rigorous estimate for the domain of validity of the method for different values of α, M, and b. However, for the case of *fixed* sparsity one can show that it is actually exact everywhere in the nonperturbative regime of the model.

Performing this straightforward but lengthy calculation and treating the discrete variable $\alpha\frac{|n-m|}{b}$ in the limit $b \gg 1$ as a continuous one, we express the LDOS $\rho_L(E - \alpha n)$ in terms of a function $G(z_1, \omega_1)$ satisfying the equation

$$\frac{\mu}{M} G(z_1, \omega_1) = \omega_1^{1/2} \int_{-\infty}^{\infty} du f\left(\frac{|u - z_1|}{\mu}\right)$$
$$\times \int_0^{\infty} \frac{d\omega}{\omega^{1/2}} e^{\frac{i}{2}\omega u - G(\omega, u)}$$
$$\times \int_0^{\infty} d\tau \tau h(\tau) J_1(\tau\sqrt{\omega\omega_1}), \quad (5)$$

where the notations $\mu = \alpha b/V$ and $z_1 = z/V$ are used. The LDOS is expressed in terms of the solution $G(z_1, \omega_1)$ as follows:

$$\rho_L(z) = \frac{1}{V}\tilde{\rho}(z/V),$$
$$\tilde{\rho}(z_1) = \frac{1}{2\pi}\text{Re}\int_0^{\infty} d\omega e^{i\omega z_1/2 - G(z_1, \omega)}. \qquad (6)$$

The pair of Eqs. (5) and (6) constitute the main analytical result of the present paper and give the expression for the LDOS of sparse WRBM in the closed form.

Let us first consider the case of the *fixed number* of nonzero matrix elements per row: $M = $ const when $b \rightarrow \infty$. At given M the form of the LDOS depends on the value of the parameter $\mu = \alpha b/V$. When this parameter is of the order of unity the LDOS form cannot be universal; i.e.; it should be dependent on the particular form of the distribution $h(\tau)$ and on M. However, the important universality emerges in the limit $\mu \gg 1$. It appears that in the large domain of ω_1, z_1 the solution of Eq. (5) can be approximately written as $G(z_1, \omega_1) = g_0\omega_1$, where g_0 is independent of z_1. Indeed, substituting this expression in the right-hand side of Eq. (5) we get

$$g_0 = \frac{\pi f(0)M}{2\mu} + O\left(\omega_1\frac{\mu}{z_1^2 + \mu^2}\right), \qquad (7)$$

which proves that our approximation is self-consistent as long as $z_1, \omega_1 \ll \mu$. Such a form of $G(z_1, \omega_1)$ immediately results in the BW form of the LDOS, Eq. (3),

in the region $|z|/V \ll \mu$, where the width Γ of the LDOS is given by

$$\Gamma = \frac{2\pi f(0)VM}{\mu} \equiv 2\pi f(0)\frac{M}{b}\frac{V^2}{\alpha}. \quad (8)$$

This allows us to give the estimate for the maximal localization length $l_{max} = \Gamma/\alpha \propto MV^2/\alpha^2 b$. As was pointed out above, our *nonperturbative* treatment (in particular, the saddle-point evaluation of the functional integral) can be valid only if $l_{max} \gg 1$. Together with the condition $\mu \gg 1$ this gives the following restriction for the region of existence of the BW regime:

$$\frac{1}{b}\sqrt{2\pi f(0)} \lesssim \frac{\alpha}{V\sqrt{M}} \lesssim \sqrt{2\pi f(0)/b}. \quad (9)$$

Another case deserving separate investigation is when the *sparsity* $S = b/M$ is fixed at $b \to \infty$ rather than M itself. In particular, the $S = 1$ case corresponds to the standard WRBM where we expect our formulas to be in agreement with those derived by Wigner [15]. Since $b \gg 1$ is equivalent for this case to $M \gg 1$, we can search for the solution of equation for $G(z_1, \omega_1)$ in terms of an expansion with respect to $1/M$. One can satisfy oneself that at the leading order in $1/M$ one can put $G(z_1, \omega_1) = 1/2g_\kappa(z_1/\sqrt{bS})\omega_1\sqrt{bS}$ where the function $g_\kappa(x)$ is the solution of the following equation:

$$g_\kappa(x) = -\frac{1}{\kappa}\int_{-\infty}^{\infty}f(|u - x|/\kappa)\frac{du}{iu - g_\kappa(u)},$$

$$\mathrm{Re}\, g_\kappa(u) < 0, \quad (10)$$

where the control parameter $\kappa = \alpha V^{-1}\sqrt{bS}$ is introduced. The LDOS is given correspondingly by

$$\rho_L(z) = \frac{1}{\pi}\mathrm{Re}\frac{1}{iz - Rg_\kappa(z/R)}, \quad R^2 = bV^2/S. \quad (11)$$

After introducing the function $r(z) = \frac{1}{\pi}[iz - Rg_\kappa(z/R)]^{-1}$ the two equations Eqs. (10) and (11) are identical to those derived by Wigner [15] provided $\alpha = 1, S = 1$, and $f(t)$ is the step function $f(t) = \theta(|t| - 1)$. It is evident that in the finite sparsity case the form of the LDOS is completely determined by the only parameter $\kappa \propto \alpha b^{1/2}$. This fact is in agreement with the mentioned numerical observation [16].

When $\kappa \ll 1$ one can immediately find from Eqs. (10) and (11) that the LDOS is given by the standard semicircular law Eq. (1) with $R_0 = (8bV^2/S)^{1/2}$ as the radius of the semicircle. When κ increases the BW form of the LDOS (3) emerges, the width Γ and the domain of validity of BW formula being given by the same expressions Eqs. (8) and (9). However, now the width Γ is actually independent of the parameter b because of $b/M = $ const. One should stress that in the BW regime the value of κ is bound because of the condition Eq. (9) which can be rewritten as $\sqrt{2\pi f(0)b} \gtrsim \kappa \gtrsim \sqrt{2\pi f(0)}$. This domain is quite narrow for not large enough values of b taken in numerical experiments [8,10].

To show the transition from the semicircle regime to that of the BW we have performed the numerical study

of Eqs. (10) and (11) together with the direct computation of the LDOS from eigenvectors of large matrices of size $N = 1000$ and band size $b = 10$ without sparsity: $S = 1$. The parameter V was kept constant in a way ensuring the semicircle radius R_0 to be unity whereas the parameter α was used to change κ. The function $f(|n - m|/b)$ was taken to be a step function: $f = 0$ for $|n - m| > b$. For such parameters $b^2 \ll N$ and the finite size corrections are small. On the other hand, there is a region for κ where the BW shape for the LDOS is expected. The results are presented in Fig. 1 where the smooth curves are solutions of Eqs. (10) and (11) and histograms are obtained by the average of Eq. (2) over the number of matrices and over those eigenvectors that are not sensitive to the finite value of N. From this figure the whole transition from the semicircle to the BW form is seen in detail. An unexpected peculiarity of these data is that in the critical point (for $\kappa \approx 2$) there is a local minimum in the center of the LDOS.

A specific question is about the form of tails of the LDOS in the BW regime [11,13,15]. It is a rather hard task to extract the analytical expression for these tails from Eqs. (10) and (11). For this reason an additional numerical check has been done for $\kappa = 8$ and $R_0 = 1$; see Fig. 2. From this figure a sharp transition from the BW to the exponential form is clearly seen at $E - \alpha n \approx \alpha b = \kappa R_0\sqrt{8} \approx 2.8$. The detailed analysis shows that it
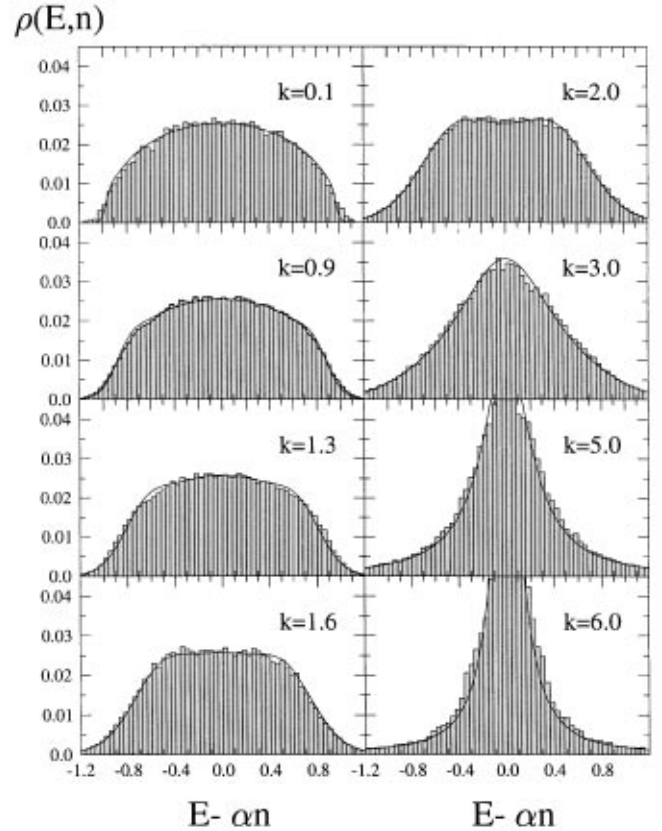

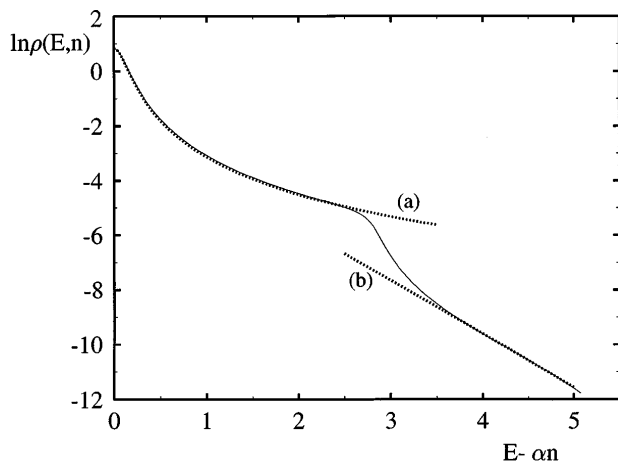FIG. 1. Dependence of the LDOS on the control parameter κ.

FIG. 2. The tail of the LDOS for $\kappa = 8$ and $R_0 = 1$. The smooth solid curve is the solution of Eqs. (10) and (11); the dotted curve (a) is the Breit-Wigner formula (3); the dotted straight line (b) is the exponential dependence.

is impossible to distinguish between the pure exponential dependence and that found in Ref. [13] where Wigner's expression for the tails [15] was corrected (see also [11]).

The results obtained here allow one to understand the nature of the *scaling* parameters governing the statistical properties of sparse WRBM in different regimes. Indeed, statistical properties of eigenvectors and eigenvalues are expected to be determined by the ratio $\beta^\star = l/l_{\max}$ where l is actual localization length in the unperturbed basis (see details in [9]). Numerical experiments [8–11] for the standard WRBM indeed revealed that the behavior of the system is completely determined by the value of the *scaling parameter* $\lambda = l_\infty/l_{\max}$ where l_∞ is the localization length l for the limit $\alpha = 0$. This parameter can also be called the "ergodicity parameter" [9] since at $\lambda \gg 1$ any eigenstate is spread uniformly over the scale $l_{\max}$ and thus occupies the maximal possible number of available states in phase space [in this case $\beta^\star(\lambda) \approx 1$].

In [22] it was found that sparsity does not actually affect the basic fact of proportionality $l_\infty \propto b^2$ at $\alpha = 0$. However, the constant $C_M = l_\infty/b^2$ is expected to be very small at small enough values of M. In contrast, in the *fixed sparsity* case $M \propto b \gg 1$ one typically has $C_M \sim 1$. In the latter situation the scaling parameter is easily shown to be $\lambda \propto (\alpha b)^2$ for the BW regime. Then the condition $\kappa \gg 1$ is equivalent to $\lambda \gg b$, immediately showing that the BW regime corresponds to the complete ergodicity, $l_\infty \gg l_{\max}$, and as a consequence to the Wigner-Dyson level statistics (see discussion in [11]). However, if the number M is fixed rather than the ratio b/M, one finds $\lambda \propto C_M(\alpha b^{3/2})^2$ for the BW domain restricted by $\lambda \gg C_M b$. For very sparse matrices the latter condition is compatible with "nonergodicity" criteria $\lambda \lesssim 1$ in view of $C_M \ll 1$. Indeed, in [12] where the extreme case $M = 1$ was studied, the level repulsion was found to be quite weak and dependent on the combination $\alpha b^{3/2}$ in agreement with the argumentation presented above.

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