Stability of Quasiequilibrium Cracks under Uniaxial Loading

Mokhtar Adda-Bedia and Martine Ben Amar
Laboratoire de Physique Statistique, Ecole Normale Supérieure, 24 rue Lhomond, F-75231 Paris Cedex 05, France*
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We propose a linear stability analysis of a straight crack subjected to uniaxial loading. We argue that, under quasistatic extension conditions, the crack propagation follows a straight path until the creation of a "physical" shear stress at its tip. This instability leads to a deviation of the fracture from the direction perpendicular to the applied loading. We compare our tip criterion instability with both experimental results and previous theoretical models.

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Recently, crack propagation problems have attracted attention of the physics community. This renewal of interest was essentially caused by experimental realizations of both equilibrium [1] and dynamic fracture mechanics [2]. From the theoretical side, the study of crack propagation can be subdivided in two classes. First, for the study of dynamical fracture formation, a long-standing problem exists. According to theory [3], cracks in brittle materials are supposed to accelerate up to the Rayleigh wave speed. In experiments, however, the cracks seldom exceed half this speed [2]. Moreover, the mechanisms that govern the dynamics of cracks are not well understood, and a theory of instability does not exist yet. The second field of crack propagation concerns slow or quasiequilibrium cracks. For this case, the work of Griffith [4] is often seen as the beginning of equilibrium fracture mechanics as a quantitative science of material behavior. Recent experiments [1] have shown that a crack traveling in a strip submitted to a nonuniform, but unidirectional, thermal diffusion field undergoes numerous instabilities. It has been established that at well-defined critical values of the control parameters a moving straight crack becomes unstable after which a wavy crack path appears. In a recent theoretical work [5], in relation with that experiment, a linear stability analysis of a straight crack based on a crack tip propagation criterion was introduced. The criterion states that the crack tip will extend out of the centered straight direction as soon as it is submitted to a "physical" shear stress. In this paper, we will show that the treatment introduced in [5], which accounts for the appearance of wavy crack patterns, is not specific to this thermoelastic problem. Moreover, we generalize this criterion to the study of the stability of a straight crack subject to any uniaxial loading in two dimensions. Although the notion of stability is systematically used for hydrodynamic systems [6], it has not been performed yet for the study of fracture problems.

As an introduction, the quasiequilibrium crack problem will be posed in its general form. Then, admitting the Griffith theory [4] and the so-called principle of local symmetry [7,8], we will perform a linear stability analysis of equilibrium cracks subjected to unidirectional loading. This defines a stability criterion for the straight crack in the presence of intrinsic perturbations due to material inhomogeneities. Finally, we will apply our approach to two simple cases and a comparison with previous results, especially those of Cotterell and Rice (CR) [8], will be done.

In a two-dimensional linear isotropic elastic model, the strain tensor $\overline{\overline{E}}(x,y)$ is related to the stress tensor $\overline{\overline{\Sigma}}(x,y)$ by a relation of the form [9]

$$E_{ij} = \frac{1}{2}\{\partial U_i/\partial x_j + \partial U_j/\partial x_i\}$$
$$= (1/2\mu)\{\Sigma_{ij} - [\kappa - 2/2(\kappa - 1)]\Sigma_{kk}\delta ij\}. \quad (1)$$

Here the subscripts are two-dimensional coordinate indices; repeated indices indicate summation. $\vec{U}$ is the displacement vector and $\kappa = 2(1-\nu)/(1-2\nu)$ [$\kappa = 2/(1-\nu)$] for a plane strain (plane stress) problem. ν is the Poisson ratio, and μ is the Lamé coefficient.

The problem of an equilibrium crack of unknown shape in an elastic medium, which is opened by tractions $-\overline{\overline{p}}(x)$, at the surface, consists of solving the equilibrium equations

$$\partial\Sigma_{ij}/\partial x_j = 0 \;\text{ and }\; \nabla^2\Sigma_{\mathrm{II}} = 0 \quad (2)$$

with the boundary conditions on the crack faces

$$(\Sigma_{ij} + p_{ij})n_j = 0\,, \quad (3)$$

where $\vec{n}$ is the unit vector normal to the crack edges. This load configuration can be either the present one or that necessary to superimpose on the stress field for an uncracked body to remove the stresses from the boundary of the crack. At this stage, we do not need to specify the conditions on the boundaries of the medium. Under equilibrium conditions, the crack shape depends essentially on the applied stresses. The mathematical formulation of this problem is as follows: Given the boundary conditions (3) for a crack whose shape is *a priori* unknown, the solution of the whole problem will determine the correct shape of the crack [9]. Formally, there might exist more than one solution to the global problem, and one has to select the crack shape which also satisfies two stability criteria.

First, the solution must satisfy a condition related to the energy criterion introduced by Griffith [4]. Defining the energy release rate G as the reduction in the total potential energy, which is the sum of the stored elastic energy W_{el} and the potential energy $-\phi$ of the external forces, associated with a small virtual crack advance ds, Griffith states that the crack is at a *critical value of incipient growth* if G is equal to the fracture energy Γ

$$G \equiv -\frac{\partial}{\partial s}(W_{\mathrm{el}} - \phi) = \Gamma\,, \text{ with } \frac{\partial G}{\partial s} \leq 0\,. \quad (4)$$

Γ is a material constant independent of the crack shape and of its dynamics.

On the other hand, the "*criterion of local symmetry*" [7,8] states that the path $y(x)$ taken by a crack in brittle homogeneous isotropic material is the one for which the local stress field at the tip is of mode I type. Let us recall that the mode I loading causes an opening of the fracture, while the mode II loading causes a shearing off. The local analysis in the neighborhood of a crack tip shows that the asymptotic stress tensor field $\overline{\overline{\Sigma}}$, in the polar coordinate system (r,θ), takes the universal form [3]

$$\Sigma_{ij}(r,\theta) = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} f_{ij}^{\mathrm{I}}(\theta) + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}} f_{ij}^{\mathrm{II}}(\theta)\,, \quad (5)$$

where $f_{ij}^{\mathrm{I}}(\theta)$ and $f_{ij}^{\mathrm{II}}(\theta)$ are universal functions common to all configurations and loading conditions. The influence of configuration and loading are included in the asymptotic description of stress only through the scalar multipliers K_{I} and K_{II}, which are the elastic stress intensity factors of the mode I and mode II loadings, respectively. The criterion of local symmetry features that, if a shear loading exists at the crack tip, $K_{\mathrm{II}} \neq 0$ and the crack will move by changing abruptly the orientation of the path.

Now consider a straight crack subjected to mode I loading and take a coordinate system so that the x axis is parallel to the crack. Nominally, $p_{ij}(x) \equiv p(x)\delta_{iy}\delta_{jy}$ and $K_{\mathrm{II}} \equiv 0$, so the criterion of local symmetry is automatically satisfied. Therefore the extension condition of the straight crack and its stability are given by Eqs. (4) only. Moreover, in this case there is a correspondence relation between the energy release rate and the stress intensity factor [10], since $G \propto K_{\mathrm{I}}^2$. In the quasistatic limit, an advancing straight crack exists if

$$K_{\mathrm{I}} = K_{\mathrm{I}c}\,, \text{ with } \partial K_{\mathrm{I}}/\partial x \leq 0\,. \quad (6)$$

Note that only positive K_{I} are permitted. If $K_{\mathrm{I}} < 0$, the crack reseals and the analysis above using vanishing traction conditions on the crack faces will not be applicable.

The question at rest is: if the conditions (6) are satisfied, does the crack always grow in the x direction for any mode I loading? In fact, due to the imperfections in the system, the stress intensity factor of mode II loading will differ slightly from zero. So, the crack alignment will be slightly perturbed [7,8]. Such a deviation, due to an instability of the straight crack, will create a shear loading which may cause the crack tip to follow a path which is amplified compared to the initial perturbation. Therefore, we investigated the problem in a different manner than in [8]: we do not analyze it at the level $K_{\mathrm{II}} = 0$, but we examine when a tiny perturbation of the path is amplified.

To perform the linear stability analysis let us introduce a small smooth perturbation, with a given wavy shape:

$$y(x) \approx Af(x) + O(A^3)\,, \quad (7)$$

where A is a constant small amplitude. The crack is arranged so that the unperturbed shape is located at $y = 0$. By small deviations from a straight crack, it must be understood that $|y(x)| \ll 1$ and $|y'(x)| \ll 1$, because the length difference between the two paths must also be small. In addition to the straight crack perturbation given by Eq. (7), we introduce a supplementary condition: $f(0) = 0$ at the crack tip (supposed to be at $x = 0$). This condition is not restrictive, since the instability occurs for a straight crack. It is therefore sufficient to compare these two configurations at the same location of the crack tip.

The perturbation method we use does not differ too much from the one followed in Ref. [8] for the study of slightly curved cracks. We develop the stress and displacement fields in A:

$$\begin{aligned} \Sigma_{ij} &= \sigma_{ij} + As_{ij} + O(A^2)\,, \\ U_i &= u_i + Av_i + O(A^2)\,, \end{aligned} \quad (8)$$

and solve first for the straight crack and then for the first order perturbation in the amplitude A. Because of the symmetry $A \to -A$, one notes that the even perturbation orders are of pure mode I type, while the odd ones are of pure mode II type. Expanding Eqs. (2) and (3) around $A = 0$, one has to solve the equilibrium equations (2) for $\overline{\overline{s}}(x,y)$, with the following conditions on the crack faces:

$$s_{yy}(x,0) = 0\,, \quad s_{xy}(x,0) = \frac{\partial}{\partial x}[f(x)\sigma_{xx}(x,0)]\,. \quad (10)$$

Using the tangential $U_t(x,y(x))$ and normal $U_n(x,y(x))$ displacements to the crack faces, one can calculate $K_{\mathrm{I}}^{\mathrm{tot}}$ and $K_{\mathrm{II}}^{\mathrm{tot}}$, the stress intensity factors of mode I and II loadings, respectively. They are [8]

$$K_{\mathrm{I}}^{\mathrm{tot}} = \alpha \lim_{x\to 0^-} \sqrt{\frac{2\pi}{-x}}\{U_n[x,y^+(x)] - U_n[x,y^-(x)]\}\,, \quad (11)$$

$$K_{\mathrm{II}}^{\mathrm{tot}} = \alpha \lim_{x\to 0^-} \sqrt{\frac{2\pi}{-x}}\{U_t[x,y^+(x)] - U_t[x,y^-(x)]\}\,, \quad (12)$$

where the superscripts $(+,-)$ design the upward and downward limits and $\alpha = \mu(\kappa - 1)/2\kappa$ is a material constant. To leading order in A, we obtain

$$K_{\mathrm{I}}^{\mathrm{tot}} = K_{\mathrm{I}} + O(A^2), \tag{13}$$

$$K_{\mathrm{II}}^{\mathrm{tot}}/y'(0) = K_{\mathrm{II}}^{*}/y'(0) + \tfrac{1}{2}K_{\mathrm{I}} + O(A^2), \tag{14}$$

which shows that $K_{\mathrm{I}}^{\mathrm{tot}}$ is still given by K_{I}, the stress intensity factor of the straight crack. The stress intensity factor K_{II}^{*} is the shear effect introduced by the first order perturbation of the loading. It is given by the resolution of a pure mode II problem for a straight crack

$$K_{\mathrm{II}}^{*} = \alpha A \lim_{x\to 0^-} \sqrt{\frac{2\pi}{-x}}\,\{v_x(x,0^+) - v_x(x,0^-)\}. \tag{15}$$

At this stage, two remarks have to be made. First, the crack will not propagate if the Griffith energy criterion is not satisfied. The stress intensity factor $K_{\mathrm{I}}^{\mathrm{tot}}$ must still satisfy Eqs. (6). Second, the linear stability analysis will not contradict the criterion of local symmetry. On the contrary, it is based on an observation which follows from this principle. When a crack is submitted to a shear loading, its extension will deviate from the pre-existing path by an angle whose sign is opposite to the one of the stress intensity factor of the mode II loading. From this observation, the stability condition for the straight crack stated hereafter follows immediately.

Our linear stability analysis is then based on the following physical arguments. If $K_{\mathrm{II}}^{\mathrm{tot}}/y'(0)$ is found positive, this means that the stress intensity factor, $K_{\mathrm{II}}^{\mathrm{tot}}$, and the orientation of the crack tip, $y'(0)$, are of the same sign. Therefore, according to the criterion of local symmetry, the crack tip tends to follow a path for which $|y'(0)|$ decreases and, consequently, the amplitude of the perturbation will decrease. On the other hand, when $K_{\mathrm{II}}^{\mathrm{tot}}/y'(0) < 0$, the slope $|y'(0)|$ will increase in order to restore a pure mode I local stress field at the tip. So, under a small perturbation of its shape, the straight crack will be stable if $K_{\mathrm{II}}^{\mathrm{tot}}/y'(0)$ is positive and unstable elsewhere. The crack path will thus deviate from a straight propagation once $K_{\mathrm{II}}^{\mathrm{tot}}/y'(0) < 0$ is satisfied. In fact, our scheme consists of searching for conditions where a small perturbation of the linear crack can create a shear loading which amplifies the intrinsic instabilities. Of course, once this threshold is reached, the extending crack will choose a curved path which satisfies $K_{\mathrm{II}}^{\mathrm{tot}} = 0$.

Although our perturbation method is close to the one of CR [8], the two stability analyses are completely different. This difference is not due to the approximation of an infinite medium assumed in [8]. More precisely, our $K_{\mathrm{II}}^{\mathrm{tot}}$ of Eq. (14) corresponds to the k_{II} of Eq. (42) in [8]. However, CR considered a straight crack which bifurcates under the effect of k_{II}. They assumed that independently of the value of k_{II} the pre-existing crack can be treated as a straight one, an assumption that we think arguable. They found that the stability condition is related to the sign of the stress in the transverse direction near the tip, which remains once the square-root singularity has been subtracted out. But, in our approach, we represented from the beginning the inhomogeneities of the material by a perturbation of the straight crack given by a fully wavy path. We analyzed when the shear effect leads to an amplification of the intrinsic deviations of the crack alignment. In fact, our stability analysis performs the approach followed in the study of a large variety of physical systems [6].

Equation (14) shows that $K_{\mathrm{II}}^{\mathrm{tot}}/y'(0)$ is the sum of two terms: The first one, $K_{\mathrm{II}}^{*}/y'(0)$, is due to the variation of the stress field $\overline{\overline{\Sigma}}$ with $\overline{\overline{\sigma}}$. The second term of Eq. (14), $K_{\mathrm{I}}/2$, is a geometrical stabilizing effect. This quantity is always positive in the range of parameters for which a straight crack can exist, so it tends to favor the straight configuration by damping the perturbation given by Eq. (7). It is foreseeable that the instability of the straight crack occurs when the perturbation of the stress field shows a destabilizing effect. That is, when $K_{\mathrm{II}}^{*}/y'(0) < 0$, which tends to amplify the instability of the straight crack. The transition will then occur when the two effects cancel exactly. This condition gives the critical values of the control parameters for which a small deviation from the straight crack begins to introduce a physical shear loading at the crack tip. At this threshold, if it exists, the straight crack becomes unstable and a curved crack path appears.

The condition for instability for the experiment described in [1] as well as the selected wavelength have been deduced quantitatively in [5]; the agreement with experiments was shown to be favorable. On the other hand, recent attempts [11] to apply the CR criterion to the same experiment has given stability thresholds significantly different from the experimental results. However, in order to examine the generality of our linear stability analysis, we will consider below two classical crack configurations in a two-bidimensional body of infinite extent which is opened by a normal mechanical traction at the surface. This configuration is chosen for its simplicity; it can be treated using Muskhelishvili's [12] method for straight cuts.

Let us take as a first example a semi-infinite crack with $p(x) = T\theta(x + l)$, where $\theta()$ is the Heaviside function and the crack tip is located at $x = 0$. The body is also assumed to be loaded at infinity by a stress RT parallel to the crack. In this case [12], $K_{\mathrm{I}} = 2T\sqrt{2l/\pi}$ and $\sigma_{xx} = [R - \theta(x + l)]T$ on the crack surface. As a simple smooth deviation of the crack shape that may exhibit the stability properties we seek, we assume that the perturbation $f(x)$ can be $f(x) = \sin \omega x$. Then Eq. (14) gives

$$\frac{K_{\mathrm{II}}^{\mathrm{tot}}}{y'(0)} = \frac{K_{\mathrm{I}}}{2}\left\{1 - R\sqrt{\frac{\pi}{2l\omega}} + \int_0^1 \frac{\cos l\omega t}{\sqrt{t}}\,dt\right\}. \tag{16}$$

Independently of the perturbation wavelength, the quantity in brackets is always positive once R is negative. So in this case, the quasistatic propagation of the straight crack is stable for $R \leq 0$. One notes that CR analysis would give a stability threshold $R = 1$.

The second problem is related to an experiment [13] on centrally cracked PMMA sheets loaded by a stress T normal to the crack and a stress RT parallel to it. For this case, one has $K_{\rm I} = T\sqrt{\pi a}$ and $\sigma_{xx} = (R - 1)T$, where $2a$ is the crack length. Using again Muskhelishvili's [12] method one finds that Eq. (14) satisfies

$$\frac{K_{\rm II}^{\rm tot}}{y'(a)} = \frac{K_{\rm I}}{2}\left\{1 - \frac{2}{\pi}(R - 1)\int_{-1}^{1}\frac{f'(at)}{f'(a)}\sqrt{\frac{1 + t}{1 - t}}\,dt\right\}. \tag{17}$$

Here the middle of the crack is chosen to be at $x = 0$. The problem is now to determine the perturbation of the straight crack path. In fact, the allowed perturbation follows from the restriction $f = 0$ at the crack extremities and from the assumption that $f(x)$ is described by a wavy smooth function. For this finite crack problem, the cases of small and large wavelength λ perturbations must be treated separately. When considering the case $\lambda \leq 2a$, the simplest form of $f(x)$ is given by $f(x) = \sin(n\pi x/a)$, with n an integer. This does not allow perturbation wavelengths larger than the crack length. Since the sought after instability is expected to derive from a large wavelength stability analysis, one must also treat the case when $f(x)$ is $f(x) = \cos\omega x - \cos\omega a$, with $\omega a \leq \pi$. If one assumes that these two forms for $f(x)$ are good representations of the perturbations, it is straightforward to show that the case of small wavelength perturbations gives a stability threshold $R = 3/2$. However, when adding the large wavelength perturbations, this threshold decreases and becomes $R = 1$. Therefore we conclude that the straight propagation of this finite crack configuration is unstable for $R > 1$ and stable for $R < 1$. This is consistent with the experimental results of [13]. Moreover, our stability threshold agrees, in this case, with that of CR [8].

In conclusion, a linear stability analysis of quasiequilibrium straight cracks subjected to an opening loading has been presented. From examination of the effect of a smooth deviation from the straight path on the various stress intensity factors, a stability criterion for the straight crack has been established. As this tip criterion compares two competitive terms which depend on the behavior of the stress field in the medium, it automatically includes the geometry and the loading conditions. This explains why in crack experiments the stability thresholds depend sensitively on the different experimental conditions. It has also been shown that the resulting stability conditions agree with both thermal [1] and mechanical [13] experiments.

The important question that remains is in what way one can extend the stability analysis to the dynamical case. To our knowledge, there is no principle equivalent to the criterion of local symmetry for a crack that is moving at velocities of the order of the Rayleigh wave speed. Attempts to find a criterion for the deviation of the dynamical straight crack tip are all related to branching instabilities [14,15]. This consists of a local analysis of the stresses near the crack tip. At this stage, criteria related to maximum stresses [14,15] in the presence (or absence) of dissipation or that of the maximum velocity allowed by the equation of motion [3] (which is an extension of the Griffith theory [4]) cannot be excluded. But since experiments often show other instabilities [2] before attaining the velocities predicted by these theories, other phenomena have to be taken into account, such as the roughness of the crack surfaces, or the acoustic emission of the crack tip.

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*Associé au CNRS, aux Universités Paris VI et Paris VII.

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