# Passive Scalars, Random Flux, and Chiral Phase Fluids

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We study the two-dimensional localization problem for (i) a classical diffusing particle advected by a quenched random mean-zero vorticity field, and (ii) a quantum particle in a quenched random mean-zero magnetic field. Through a combination of numerical and analytical techniques we argue that both systems have extended eigenstates at a special point in the spectrum invariant under particle-hole symmetry, where a sublattice decomposition obtains. In a neighborhood of this point, the Lyapunov exponents of the transfer matrices acquire ratios characteristic of conformal invariance allowing an indirect determination of $1/r$ for the typical spatial decay of eigenstates.

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In this paper we study two simple models for passive advection of a diffusing field: (I) a diffusing scalar density $n(\mathbf{x})$ advected by a quenched random velocity field $\mathbf{A}(\mathbf{x})$ described by the Fokker-Planck equation [1]:

$$\partial_t n = \mathcal{L}_{fp} n \equiv D\nabla^2 n - \boldsymbol{\nabla}\cdot(\mathbf{A}n)\,, \qquad \text{(I)}$$

where $D$ is the diffusivity; and (II) the random-flux model [2] for a noninteracting quantum particle propagating in a spatially random, zero-mean magnetic field $B \equiv \boldsymbol{\nabla}\times\mathbf{A} \equiv \partial_x A_y - \partial_y A_x$, where $\mathbf{A}$ now denotes the vector potential, described by the Schrödinger equation

$$-i\partial_t\psi = \mathcal{L}_{rf}\psi \equiv \{(\mathbf{p}-\mathbf{A})^2 + V\}\psi\,, \qquad \text{(II)}$$

$\psi$ being the (complex) quantum wave function, $\mathbf{p} \equiv -i\boldsymbol{\nabla}$ the momentum operator, and $V(\mathbf{x})$ the (scalar) potential [3].

Model (II) has received much attention recently in the context of the quantum Hall effect at filling factor $\nu = \frac{1}{2}$ [2]. An unresolved question is whether the system has properties of a Fermi liquid, and, in particular, extended states. Previous work has addressed the energy dependence of the localization length $\xi(E)$ moving inward from the band edge, with authors arriving at opposite conclusions. The most careful numerical study of model (II) to date concludes that all states are localized [2(b)], whereas others find a central *band* of extended states [2(a)]. An analytic calculation using a replicated nonlinear sigma model with a topological term [4] also obtains a band of extended states.

Our purpose is twofold. First, we explore the consequences of particle-hole symmetry at the band center of these models, $E_c$, and describe numerical and analytical evidence for a divergent localization length at this point. Previous studies [2(b),5] did not allow for this symmetry at the band center [6]. Second, we demonstrate that the properties of random flux that have drawn so much attention are exhibited by a much larger class of models, among them the passive scalar model (I).

Magnetic field and vorticity are distinguished from potential fields by their transformation under time reversal [7]. Writing the velocity in model (I) as $\mathbf{A} = \boldsymbol{\nabla}\chi + \boldsymbol{\nabla}\times\phi$, we observe that $\chi$, like $V$, is even under time reversal, whereas $\phi$, (the source of vorticity $\omega = \boldsymbol{\nabla}\times\mathbf{A} = -\nabla^2\phi$), like $B$, is odd. A further physical similarity between the two models is that one expects transport to be dominated by the longest streamlines [1(b)] [8,9]; for the random-flux (passive scalar) model with vanishing mean magnetic field (vorticity), these rare streamlines run along the interfaces of opposing magnetic field (vorticity). The Laplacian enables fields to tunnel (diffuse) among distinct closed streamlines, and creates competition between *advection* by streamlines that can transport the field over long distances without attenuation, and *diffusion* that leads to destructive interference [10].

We study spatial decay of the eigenfunctions for lattice approximations to $\mathcal{L}_{fp}$ and $\mathcal{L}_{rf}$. $\mathcal{L}_{fp}$ is not self-adjoint, and when $\phi \neq 0$ its eigenvalues $z$ occupy, in general, an *area* in the complex plane. For given $z$ we use well-established numerical transfer-matrix methods and finite-size scaling [11] to compute the localization length [12] on a long strip.

The real scalar field $n$ is discretized on a square lattice of width $L$, length $m$. Boundary conditions in the transverse ($L$) direction will be discussed later. $n$ and $\chi$ are defined on nodes, $\mathbf{A}$ on links, and $\phi$ on the nodes of the dual lattice. We define lattice difference operators $\Delta^+_\mu\phi_{\mathbf{x}} \equiv \phi_{\mathbf{x}+\mathbf{e}_\mu} - \phi_{\mathbf{x}}$ and $\Delta^-_\mu\phi_{\mathbf{x}} \equiv \phi_{\mathbf{x}} - \phi_{\mathbf{x}-\mathbf{e}_\mu}$, where $\mathbf{e}_{\mu=x,y}$, are orthogonal lattice basis vectors. The velocity is then $\mathbf{A} = \Delta^-\times\phi + \Delta^+\chi$, where $\Delta^-\times\phi \equiv (-\Delta^-_y\phi, \Delta^-_x\phi)$ represents the discrete curl. Defining the current $\mathbf{J} = -D\Delta^+ n + \mathbf{A}n$, where $[\mathbf{A}n]^\mu_{\mathbf{x}} \equiv \frac{1}{2}A^\mu_{\mathbf{x}}[n_{\mathbf{x}} + n_{\mathbf{x}+\mathbf{e}_\mu}]$ represents an average of the two ends of the link, the equation of motion is $\partial_t n + \Delta^-\cdot\mathbf{J} = 0$.

Similarly, to discretize the random-flux Hamiltonian we define the lattice covariant derivative:

$$D^+_\mu\psi_{\mathbf{x}} \equiv e^{i\mathbf{A}^\mu_{\mathbf{x}}}\psi_{\mathbf{x}+\mathbf{e}_\mu} - \psi_{\mathbf{x}}\,,$$
$$\mathcal{L}_{rf} \equiv D^+_\mu D^-_\mu + V_{\mathbf{x}}\,.$$

  

The vector potential $\mathbf{A}$ has been defined on links in the same way as for the fluid; the scalar potential $V_{\mathbf{x}}$ vanishes unless otherwise stated.

Values of the field can be computed recursively using the $2L \times 2L$ transfer matrices $W_k$, which yield values of $\psi$ or $n$ in lattice column $k+1$ and $k$ given those in columns $k$ and $k-1$. Lyapunov exponents are extracted as logarithms of the $2L$ eigenvalues of the matrix $(W^{(m)\dagger} W^{(m)})^{1/2m}$ in the limit $m \to \infty$, where $W^{(m)} \equiv \prod_{k=1}^{m} W_k$; correlation lengths along the strip then correspond to their inverses. We define the scaled localization length by $\xi_L(z) = 1/\lambda_L(z)L$, where $\lambda_L(z)$ is the exponent smallest in magnitude. A critical or extended phase occurs for those $z$ where $\xi_L(z) \to \xi_\infty(z) \neq 0$ for large $L$.

We first describe results for model (I) for incompressible $\mathbf{A} = \Delta^- \times \phi$, taking the $\phi_{\mathbf{x}}$ to be independent random variables distributed uniformly over an interval $[-w, w]$. When $\phi \equiv 0$ the eigenvalues fill (uniformly, in $d = 2$) the real interval $[-8D, 0]$. For nonzero $\phi$ the density of states broadens into a complex neighborhood of this interval. Figure 1 displays $\xi_L(z)$ using $D = 1/4$, $w = 1$, $L = 32$, and periodic boundary conditions. The peaks at $z = 0, -8D$ arise because the (extended) eigenfunctions, corresponding, respectively, to $n(\mathbf{x}) = n_0$ and to uniform antiferromagnetic $n(\mathbf{x})$, represent exact solutions for any width. The structure is symmetric about the line $\mathrm{Re}\,(z) = E_c \equiv -4D$, a feature that originates in an exact *particle-hole* symmetry of $\mathcal{L}_{fp}$. For $L$ *even*, we divide the lattice into its two equivalent antiferromagnetic sublattices, with $n_+$ and $n_-$ the restriction of $n$ to the two sublattices, and $\hat{\lambda} \equiv \binom{n_+}{n_-}$. We can now express $\mathcal{L}_{fp}$ in block form operating on $\hat{\lambda}$:

$$\mathcal{L}_{fp} = \begin{pmatrix} E_c I_L & \mathcal{Q} \\ \tilde{\mathcal{Q}} & E_c I_L \end{pmatrix}, \tag{1}$$

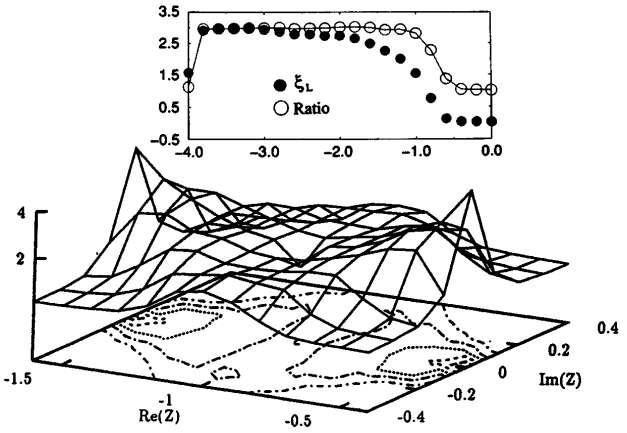


FIG. 1. (a) Scaled localization length $\xi_L(z)$ for model (I), and contour plot. Parameters are $D = \frac{1}{4}$, $w = 1$, $L = 32$, with periodic boundary conditions. The inset shows $\xi_L(z)$ for model (II), with $L = 32$ and fluxes chosen independently on each plaquette and uniformly on $[0, 2\pi]$ (filled circles). Empty circles display the ratio $\lambda_L^2/\lambda_L^1$. Statistical error in $\xi_L(z)$ is $\simeq 5\%$.

where $I_L$ is the $L \times L$ unit matrix, $\tilde{\mathcal{Q}} = T\mathcal{Q}^\top T^{-1}$, $\top$ denotes transposition, and $T$ time reversal. This transformation inverts the sign of the antisymmetric parts of $\mathcal{L}_{fp}$. The symmetry may be stated as follows: If $\hat{\lambda}_z$ is an eigenvector of $\mathcal{L}_{fp}$ with eigenvalue $z$, then $\hat{\lambda}_{\bar{z}} \equiv \sigma_z \hat{\lambda}^* \equiv \binom{n_+^*}{-n_-^*}$ is an eigenvector with eigenvalue $\bar{z} = 2E_c - z^*$. The same symmetry applies to the random flux operator, where $E_c = 4$, $z$ is real, and $\mathcal{Q}^\dagger = \tilde{\mathcal{Q}}$.

At $z = E_c$ the two sublattices decouple, and the eigenvectors of interest correspond to the zero eigenvalues of $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$. Furthermore, in the limit $m \to \infty$, the Lyapunov exponents at $E_c$ must occur in degenerate pairs; this degeneracy is obtained numerically and disappears for any $z \neq E_c$. The decoupling is also the source of the striking depression in $\xi_L$ at $E_c$ seen in Fig. 1 for both models.

The depression and the degeneracy occur only for even $L$. For odd $L$, periodic boundary conditions mix the two sublattices; $\xi_L(E)$ reaches (at $E \simeq -2.0$ for the random flux model; see inset to Fig. 1) a plateau of twice its degenerate value as one moves in from the band edges, and maintains that value at the band center. For odd $L$ and *free* boundary conditions (as used in some numerical studies [2(a)]), $\xi_L(E_c)$ *diverges* as $m \to \infty$ for any finite $L$. We obtain both degeneracy and divergence also for the "$q$" models [2(a)], where the fluxes are restricted to values $2\pi n/q$ with $q, n$ integers. These properties have not been identified before.

Numerical evidence alone cannot distinguish an infinite localization length from a large but finite one. We now offer analytic arguments in support of a divergent correlation length at $E_c$ for the random-flux model with free boundary conditions. Let $\hat{n}_k \equiv \binom{n_k}{n_{k-1}}$ represent the $2L$-component vector composed of the $n_{\mathbf{x}}$ in columns $k$ and $k-1$. With an appropriate choice of gauge, we can write the transfer matrix in the form $W_k \equiv \binom{\Theta_k \ -I_L}{I_L \ \ 0}$, where $\Theta_k$ is Hermitian. Observe that $W_k^\dagger J W_k = J$, where $J \equiv \binom{0 \ -I_L}{I_L \ \ 0}$; the set of matrices satisfying this identity constitutes a group. It follows that the eigenvalues of $W_k$ occur in inverse conjugate pairs, $\mu, 1/\mu^*$, as do the eigenvalues of any product of $W_k$'s.

The sublattice decomposition enables us to reorganize the components of $\hat{n}_k$ in the form $\hat{n}_k = \mathrm{col}\,\{n_k^+, n_{k-1}^+, n_k^-, n_{k-1}^-\}$. If $L = 2l+1$ is odd, the number of components of $n_k^\pm$ will alternate with $k$ between $l$ and $l+1$. The transfer matrix now takes the block form $W_k = \binom{w_k \ \ 0}{0 \ \ \tilde{w}_k}$, where $w_k = \binom{\theta_k \ -I}{I \ \ 0}$ and $\tilde{w}_k = \binom{\theta_k^\dagger \ -I}{I \ \ 0}$. This new form of the transfer matrix only connects sites on the same sublattice, and the $\theta_k$ are no longer Hermitian nor (for odd $L$) necessarily square.

Because the ensemble of random matrices weights $\theta_k$ and $\theta_k^\dagger$ equally, we expect that the eigenvalues of the submatrix products $(w^{(m)\dagger} w^{(m)})^{1/2m}$ and $(\tilde{w}^{(m)\dagger} \tilde{w}^{(m)})^{1/2m}$ are identical in the limit $m \to \infty$, where $w^{(m)} \equiv \prod_{k=1}^{m} w_k$ and $\tilde{w}^{(m)} \equiv \prod_{k=1}^{m} \tilde{w}_k$. It follows that the eigenvalues of the full transfer-matrix product occur in degenerate pairs; this degeneracy is observed numerically. Since

the full transfer-matrix product is a group element, we deduce that for *odd* $L$ there must be a pair of eigenvalues with modulus unity, one from each of the two submatrix products, yielding a divergent $\xi_L(E_c)$.

For *even* $L$, an eigenvalue of modulus unity is not expected for finite $L$, and we instead argue that a pair of eigenstates exists at $E_c$ in the thermodynamic limit. To make further progress we turn our attention from the transfer matrix $W$ to the operator $\mathcal{L}_{rf}$ itself, and exploit its off-diagonal block form (1) at $E_c$. (For convenience, we translate the 0 of energy to $E_c$ in this discussion.) For a lattice with an odd number of sites, $N$, $\mathcal{Q}$ is not square so that it has a nontrivial kernel and 0 is an eigenvalue of $\mathcal{L}_{rf}$. As a consequence of the singular value decomposition, adding a new lattice site (reversing the parity of $N$) can never increase the magnitude of the smallest nonzero eigenvalue. Because the randomness in $\mathcal{Q}$ can be expected to remove any accidental exact or near degeneracy, we anticipate rather that the magnitude of the smallest eigenvalue above 0 (and its particle-hole conjugate below 0) diminishes as $N \to \infty$, yielding a degenerate pair of 0 eigenvalues.

If there are indeed two independent eigenfunctions at 0 in the limit $N \to \infty$ through even values, they must take the form $\binom{u}{\pm v}$, $u, v$ the left, right eigenvectors of $\mathcal{Q}$ so that $\mathcal{Q}^\dagger u = \mathcal{Q} v = 0$. Because $\langle u | v \rangle \neq 0$, in general, we see that the arbitrary relative phase of $u$ and $v$ implies a continuous $GL(1, C)$ symmetry at $E_c$. Such a continuous symmetry is known to play an important role in some closely related random-matrix models. Wegner first observed the significance of the sublattice decomposition in a class of random-matrix models for localization in $d = 2$ [13]. It was later noticed that the sublattice decomposition allows a new continuous symmetry, which contains in the $n = 0$ replica limit a factor of U(1). For lattice models with spin, this continuous symmetry leads to a divergent density of states (DOS) and localization length [13]. Our numerics indicate a finite DOS at $E_c$, a result that is not, in general, inconsistent with a divergent correlation length. We are pursuing an analogous replicated field theory calculation for our model; results so far are consistent with the existence of this symmetry [10].

We turn now to the *neighborhood* of the band center. Assuming finite-size scaling, we expect $\xi_L(z)$ to be independent of $L$ (though not of $z$; see [14]) as $L \to \infty$ in the regime of extended states. In Fig. 2, we show $\xi_L(z)$ for several values of $z$ and $L$. Close to $E_c$, numerical values for $\xi_L(z)$ are more or less independent of $L$ when $z$ is on the lines Im $(z) = 0$ or Re $(z) = E_c$.

For $E_c$ and its neighborhood we have examined the entire Lyapunov spectrum, and find that after appropriate scaling (see Fig. 3) the spectra for distinct $L$ collapse onto a single curve. In addition, for small $n/L$, ratios of Lyapunov exponents take the form

$$\lambda_L^n / \lambda_L^1 = 2n - 1 , \qquad (2)$$

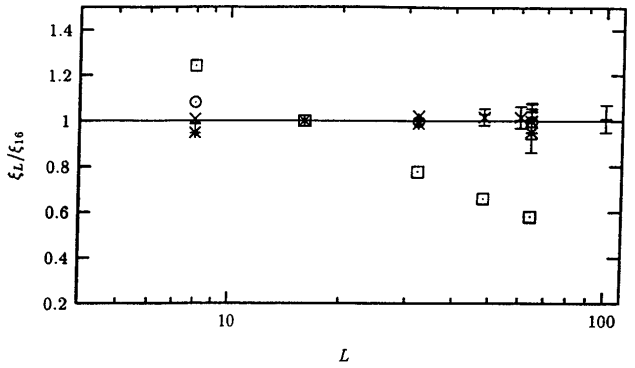


FIG. 2. Scaled localization length $\xi_L(z)/\xi_{16}(z)$ plotted as a function of $L$ for various $z$. $+$, $\times$, square, *, and circle represent, respectively, $z = -1.0, -1.0 + 0.25i, -1.0 + 0.6i, -1.25, -1.25 + 0.25i$. Error bars are displayed when they exceed the symbol width.

where $\lambda_L^j$ denotes the $j$th largest positive Lyapunov exponent. This form obtains to both models (I) and (II) even when continued to complex **A** (this allows an approximate interpolation between the two models), for random independent $\mathbf{A}_{\mathbf{x}}^{\mu}$, and also when the $B_{\mathbf{x}}$ (fluxes) in model (II) are taken to be independent random variables. As explained elsewhere [14], the form $R_n \equiv (n + x)/x$ for the ratio of Lyapunov exponents suggests that the single-particle Green's function is conformally invariant for a typical realization of the disorder, and typically decays as $1/r^\eta$, where $\eta = 2x$. Evidently, for our models $\eta \simeq 1$.

So far we have discussed only divergence-free **A**. For model (I), the addition of a random $\chi$ leads to a real nonzero diagonal component of the discretized model at $E_c$ (as does scalar potential disorder $V_{\mathbf{x}}$ for model (II)—note that $\chi$ may always be removed by a gauge transformation). The diagonal components of the disorder invalidate exact particle-hole symmetry and the sublattice decomposition. When the disorder amplitude for the divergence-free field sufficiently exceeds that of the curl-free component, our numerics are consistent with extended states, and the form $x \simeq 1/2$ for the ratio survives. In the opposite case, for large enough $w$, we find that all states in the neighborhood of $E_c$ are localized.

We have done further numerical computations in order to isolate the property of the operators $\mathcal{L}_{fp}$ and $\mathcal{L}_{rf}$ responsible for the apparent band of extended states [10]. It is found that symmetric real random-matrix models with nearest-neighbor couplings and a sublattice decomposition display at $E_c$ degenerate extended states and $x \simeq 1/2$, but away from $E_c$ all states are unambiguously localized, as is observed for the $q = 2$ model [2(a)], and $R_n$ has no special form. The addition of a random, uncorrelated antisymmetric matrix (real or complex) produces, in the neighborhood of $E_c$, a band of states for which the Lyapunov ratios satisfy $x \simeq 1/2$ and $\xi_L$ has no detectable dependence on $L$. Recalling that the *symplectic* group

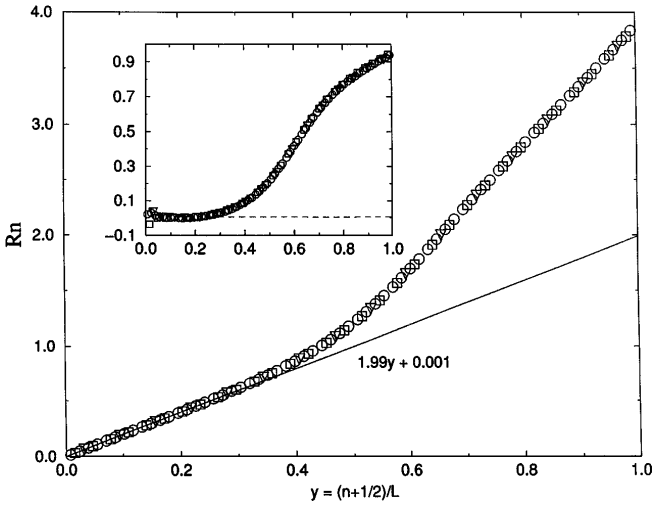


FIG. 3. Comparison of the ratios, $R_n = \lambda_n/\lambda_{n_0}$, for various different $L$ with Eq. (2) (dashed line). The largest correlation length corresponds to $n = 0$. A linear fit for small $n/L$ yields $x = 0.50 \pm 0.02$. To minimize the error in $R_n$ we choose a small $n_0$ for which the error is less than 1%. We collapse the ratios for different $L$ by defining $y = (n + 1/2)/L$ and rescaling the ratio by $(2n_0 + 1)/L$. Parameters are $z = -1.0 + 0.25i$, $D = \frac{1}{4}$, and $n_0 = 8, 4, 2$ for $L = 64$ (circle), 32 (square) and 16 (triangle), respectively. For $z = -1.0$ a similar picture is obtained with $x = 0.49 \pm 0.02$. The inset shows, for the same data set, the relative difference between the numerically obtained values and Eq. (2). The dashed line displays the statistical error.

constitutes the set of transformations under which a bilinear antisymmetric form is invariant, we conjecture that the usual assumption that this class of model has merely *unitary* symmetry [2(b),4,5] may not be justified.

We propose that *chiral phase fluids*, systems with divergent correlation lengths originating in a random antisymmetric contribution to their dynamics, are a general phenomenon and share universal features. A full understanding of these systems must answer the question, not so far resolved numerically, of whether the only extended states are in fact at the band center, and if so, what sets the scale of the correlation length in the remainder of the band. Since our arguments are general in nature, we expect to see similar universal behavior in other systems, among them the kinematic dynamo [15].

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