Surface Critical Phenomena and Scaling in the Eight-Vertex Model

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We give a physical interpretation of the entries of the reflection K matrices of Baxter's eight-vertex model in terms of an Ising interaction at an open boundary. Although the model still defies an exact solution, we nevertheless obtain the exact surface free energy from a crossing-unitarity relation. The singular part of the surface energy is described by the critical exponents $\alpha_s = 2 - \pi/2\mu$ and $\alpha_1 = 1 - \pi/\mu$, where μ controls the strength of the four-spin interaction. These values reduce to the known Ising exponents at the decoupling point $\mu = \pi/2$ and confirm the scaling relations $\alpha_s = \alpha_b + \nu$ and $\alpha_1 = \alpha_b - 1$.

Our understanding of phase transitions and critical phenomena has been greatly enhanced by the study of exactly solved lattice models in statistical mechanics [1]. Chief among these models is Baxter's eight-vertex model, which exhibits continuously varying critical exponents [2]. Such exact results provide valuable insights into the key theoretical developments of universality, renormalization, and scaling. The eight-vertex model is equivalent (see Figs. 1 and 2) to two Ising models coupled together by four-spin interactions [3,4]. From [1–4] the singular part of the bulk free energy f_b scales as $f_b \sim |t|^{\pi/\mu}$ as $t \to 0$. Here t vanishes linearly with $T - T_c$, where T_c is the critical temperature. The variable μ measures the strength of the four-spin interaction M via $\exp(2M) = \tan(\mu/2)$. When $\mu = \pi/m$, where m is an even integer, the critical behavior is modified to $f_b \sim |t|^{\pi/\mu} \log|t|$. This is the case in the Ising limit, where $\mu = \pi/2$. The critical exponent describing the divergence of the bulk specific heat, $C_b \sim |t|^{-\alpha_b}$ as $t \to 0$, is given by $\alpha_b = 2 - \pi/\mu$, with $\alpha_b = 0$ (log) for the Ising model.

A significant test of the scaling relations between critical exponents was given by Johnson, Krinsky, and McCoy [5], who derived the correlation length exponent $\nu = \pi/2\mu$ for the eight-vertex model. Together with Baxter's result for α_b this confirmed the validity of the bulk scaling law [1,6,7] $\alpha_b = 2 - 2\nu$. However, the situation is not so satisfactory for the *surface* critical behavior [6,7], as the eight-vertex model has not been solved for *open* boundary conditions as in Fig. 2. Whereas integrability in the bulk is governed by solutions of the Yang-Baxter equation [2,8,9], integrability in the presence of boundaries is governed by solutions of both the Yang-Baxter and reflection equations [10,11]. K matrices satisfying the reflection equations have been found for the eight-vertex model [12–15], but the diagonalization of the transfer matrix remains a formidable problem. Here we nevertheless derive two surface critical exponents of the eight-vertex model, allowing a direct test of the proposed scaling relations between bulk and surface critical exponents [6,7,16].

The relation between the bulk Boltzmann weights a, b, c, d of the eight-vertex model and the Ising couplings K, L, M is depicted in Fig. 1. These weights are given by [1]

$$a(u) = \rho_0 \theta_4(\lambda)\theta_4(u)\theta_1(\lambda - u),$$

$$b(u) = \rho_0 \theta_4(\lambda)\theta_1(u)\theta_4(\lambda - u),$$

$$c(u) = \rho_0 \theta_1(\lambda)\theta_4(u)\theta_4(\lambda - u),$$

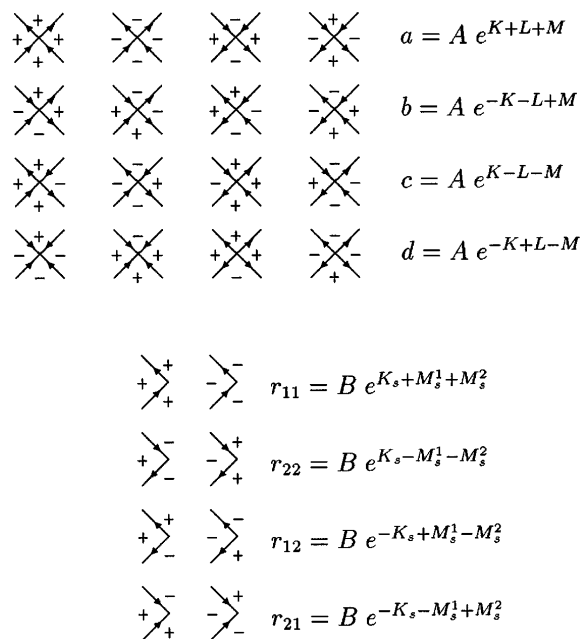$$d(u) = \rho_0 \theta_1(\lambda)\theta_1(u)\theta_1(\lambda - u), \tag{1}$$

FIG. 1. The bulk and surface vertex and Ising spin configurations and their corresponding Boltzmann weights. The nearest-neighbor bulk interactions K and L are in the vertical horizontal directions, respectively. The four-spin interactions is denoted by M and the general nearest-neighbor surface interactions by K_s, M_s^1 and M_s^2 in the vertical, SW-NE, and SE-NW directions, respectively. The constants A and B do not enter into the critical properties.
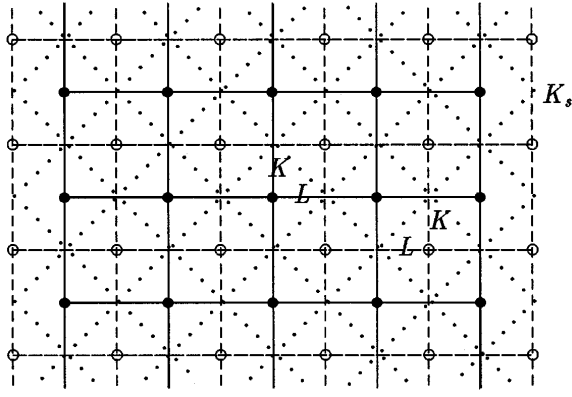
FIG. 2. The geometric relation between the eight-vertex model lattice (dotted lines) and the Ising model lattice (broken and solid lines). The Ising lattice is divided into two sublattices (solid and open circles).

where ρ_0 is a normalization factor. Here $\theta_1(u) = H(u)$ and $\theta_2(u) = \Theta(u)$ are the elliptic theta functions, of nome $q = \exp(-\pi I'/I)$, where I and I' are the half-period magnitudes [1]. In the principal regime, $0 < u < \lambda$, with $0 < \lambda < I'$ and $0 < q < 1$. Here $q \to 1$ at criticality. In terms of the vertex weights, the bulk Ising couplings are given by

$$\exp(4K) = \frac{ac}{bd} = \left[\frac{\theta_4(u)}{\theta_1(u)}\right]^2, \tag{2}$$

$$\exp(4L) = \frac{ad}{bc} = \left[\frac{\theta_1(\lambda - u)}{\theta_4(\lambda - u)}\right]^2, \tag{3}$$

$$\exp(4M) = \frac{ab}{cd} = \left[\frac{\theta_4(\lambda)}{\theta_1(\lambda)}\right]^2. \tag{4}$$

In the Ising limit, $M = 0$ when $\lambda = \frac{1}{2}I'$, with the spectral variable u controlling the anisotropy of the Ising couplings K and L.

For the lattice orientation of Fig. 2, the integrable boundary vertex weights can be written down from the entries of the K matrix. Now for the eight-vertex model this reflection matrix is 2×2, of the general form $K^-(u) = K(u; \xi_-, \eta_-, \tau_-)$, with elements [15]

$$K_{11} = \rho_s \frac{\theta_1(\xi - u)}{\theta_4(\xi - u)}, \qquad K_{22} = \rho_s \frac{\theta_1(\xi + u)}{\theta_4(\xi + u)}, \tag{5}$$

$$K_{12} = \rho_s \eta \theta_4^2(\xi) \frac{\theta_1(2u)}{\theta_4(2u)}$$

$$\times \frac{\{\tau[\theta_4^2(u) + \theta_1^2(u)] - \theta_1^2(u) + \theta_4^2(u)\}}{\theta_4^2(\xi)\theta_4^2(u) - \theta_1^2(\xi)\theta_1^2(u)}, \tag{6}$$

$$K_{21} = \rho_s \eta \theta_4^2(\xi) \frac{\theta_1(2u)}{\theta_4(2u)}$$

$$\times \frac{\{\tau[\theta_4^2(u) + \theta_1^2(u)] + \theta_1^2(u) - \theta_4^2(u)\}}{\theta_4^2(\xi)\theta_4^2(u) - \theta_1^2(\xi)\theta_1^2(u)}, \tag{7}$$

Here ρ_s is a normalization factor and ξ, η, τ are arbitrary parameters. In principle, these three parameters are related to the surface couplings. We argue that the variable ξ controls the strength of the Ising surface coupling K_s. Similar to the bulk case, we see from Fig. 1 that $\exp(4K_s)$ is given by a ratio of the boundary weights r_{ij}, which in turn are related to the K-matrix elements [17],

$$\exp(4K_s) = \frac{r_{11}r_{22}}{r_{12}r_{21}} = \frac{K_{11}(u/2)K_{22}(u/2)}{K_{12}(u/2)K_{21}(u/2)}. \tag{8}$$

The particular choice $\tau = 0$ and $\xi = \frac{1}{2}I'$ simplifies to

$$\exp(4K_s) = -\frac{1}{\eta^2}\left[\frac{\theta_4(u)}{\theta_1(u)}\right]^2. \tag{9}$$

Comparison with the bulk parametrization (2) implies that the further choice of $\eta^2 = -1$ leads to $K = K_s$, i.e., equal bulk and surface couplings in the Ising spin formulation. These particular values of η and τ can be chosen for all ξ, since the surface coupling K_s can be clearly set to be independent of η and τ, which can be seen from the product $r_{11}r_{22}$.

The surface free energy can be obtained by applying the inversion relation method, which is known to give the correct bulk free energy of the eight-vertex model (see, e.g., Sec. 13.7 of Ref. [1]). By using the fusion procedure, the transfer matrix of the eight-vertex model with boundaries described by $K^\pm$ matrices has recently been found to satisfy a group of functional relations [18,19]. Ignoring the finite-size corrections, which are of no relevance here, the relations give the desired crossing-unitarity relation for the transfer matrix eigenvalues [20],

$$\Lambda(u)\Lambda(u + \lambda) = \omega_+(u)\omega_-(u)\rho^{2N}(u). \tag{10}$$

The factor

$$\rho(u) = \frac{\theta_1(\lambda - u)\theta_1(\lambda + u)}{\theta_1(\lambda)\theta_1(\lambda)} \tag{11}$$

is a bulk contribution whereas the product $\omega_+(u)\omega_-(u)$ is a surface contribution, with [18,19]

$$\omega_+(u) = K_{11}^+(u)b(-2u + \lambda)K_{22}^+(u + \lambda) + K_{12}^+(u)d(-2u + \lambda)K_{12}^+(u + \lambda)$$

$$- K_{21}^+(u)a(-2u + \lambda)K_{12}^+(u + \lambda) - K_{22}^+(u)c(-2u + \lambda)K_{22}^+(u + \lambda), \tag{12}$$

$$\omega_-(u) = K_{21}^-(u + \lambda)d(2u + \lambda)K_{21}^-(u) + K_{22}^-(u + \lambda)b(2u + \lambda)K_{11}^-(u)$$

$$- K_{11}^-(u + \lambda)c(2u + \lambda)K_{11}^-(u) - K_{12}^-(u + \lambda)a(2u + \lambda)K_{21}^-(u). \tag{13}$$

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Here $K^+(u)$ is the transpose of $K^-(-u + \lambda)$ with ξ_- replaced by ξ_+, etc.

The bulk and surface free energies must both satisfy the crossing-unitarity relation (10). The surface energy can be separated from the bulk energy. As we are only predominantly interested here in the surface critical behavior rather than the precise form of the surface energy, we consider only the diagonal elements of the K matrix. These terms are sufficient to extract the critical exponents and

physically we do not anticipate any change in the critical behavior arising from the off-diagonal terms [21]. Define $\Lambda_b = \kappa_b^{2N}$ and $\Lambda_s = \kappa_s$, then the bulk and surface free energies per site are defined by $f_b(u) = -\ln \kappa_b(u)$ and $f_s(u) = -\ln \kappa_s(u)$. From (10)–(13) we have

$$\kappa_b(u)\kappa_b(u + \lambda) = \rho(u) \tag{14}$$

for the bulk and

$$\kappa_s(u)\kappa_s(u + \lambda) = \frac{\theta_1(\xi_- - u)\theta_1(\xi_- + u)\theta_1(\xi_+ - u)\theta_1(\xi_+ + u)}{\theta_1^4(\lambda)} \frac{\theta_1(2\lambda - 2u)\theta_1(2\lambda + 2u)}{\theta_1^2(2\lambda)} \tag{15}$$

for the surface.

We obtain the solution of (15) for $\kappa_s(u)$ by applying the inversion relation method [1]. Let us first recall the derivation of $\kappa_b(u)$ from (14). It is convenient to introduce the variables $x = \exp(-\pi\lambda/2I)$ and $w = \exp(-\pi u/I)$. To obtain $f_b(w)$ the argument is to assume that $\kappa_b(w)$ is analytic and nonzero in the annulus $x^2 \leq w \leq 1$, allowing the Laurent expansion of $f_b(w)$,

$$\ln \kappa_b(w) = \sum_{n=-\infty}^{\infty} c_n w^n. \tag{16}$$

Inserting this expansion into the logarithm of both sides of (14) and equating the coefficients of powers of w then gives

$$f_b(w) = -\sum_{n=1}^{\infty} \frac{(x^{2n} + q^{2n}x^{-2n})(1 - w^n)(1 - x^{2n}w^{-n})}{n(1 + x^{2n})(1 - q^{2n})}. \tag{17}$$

This is the desired result, from which the critical behavior in the limit $q \to 1$ is extracted by use of the Poisson summation formula [1]. In terms of the variable $\mu = \pi\lambda/I'$, where $I' \to \pi/2$ as $q \to 1$, it follows that $f_b \sim p^{\pi/\mu}$ as $p \to 0$, with $f_b \sim p^{\pi/\mu} \ln p$ if π/μ is an even integer. Here the conjugate nome $p = \exp(-2\pi I/I')$ vanishes linearly with the deviation from criticality variable t [1].

We obtain the surface free energy by solving (15) under the *same* analyticity assumptions as for the bulk case, together with the further assumption that $\kappa_s(w)$ is analytic and nonzero in the annulus $x < y_\pm < 1$, where we have defined $y_\pm = \exp(-\pi\xi_\pm/2I)$. In this way we arrive at the result

$$f_s(w, y_\pm) = \sum_{n=1}^{\infty} \frac{(y_+^{2n} + q^{2n}y_+^{-2n} + y_-^{2n} + q^{2n}y_-^{-2n})(w^n + x^{2n}w^{-n})}{n(1 + x^{2n})(1 - q^{2n})}$$

$$- \sum_{n=1}^{\infty} \frac{(x^{4n} + q^{2n}x^{-4n})(1 - w^{2n})(1 - x^{4n}w^{-2n})}{n(1 + x^{4n})(1 - q^{2n})}. \tag{18}$$

Applying the Poisson summation formula leads to a series for f_s in powers of the nome p.

The phenomenology of critical behavior at a surface is well developed [6,7,16]. In this case two surface critical exponents can be obtained from the surface free energy; one from the surface specific heat, $C_s \sim |t|^{-\alpha_s}$, and the other from the "local" specific heat in the boundary layer, $C_1 \sim |t|^{-\alpha_1}$. Here the corresponding surface internal energy is given by

$$e_s(p) \sim \frac{\partial f_s(u, \xi_\pm)}{\partial p} + e_1(p), \tag{19}$$

where $e_1(p)$ is the first layer internal energy,

$$e_1(p) \sim \frac{\partial f_s(u, \xi_\pm)}{\partial \xi_\pm}. \tag{20}$$

The related specific heats follow as

$$C_s \sim \frac{\partial e_s}{\partial p} \quad \text{and} \quad C_1 \sim \frac{\partial e_1}{\partial p}. \tag{21}$$

These definitions follow from [6,7,16] with the identifications $p \sim t$ and $\xi_\pm \sim K_s$. From (18) we find that as $p \to 0$

$$e_s(p) \sim p^{\pi/2\mu - 1} \quad \text{and} \quad e_1(p) \sim p^{\pi/\mu}. \tag{22}$$

As for the bulk case, a logarithmic factor appears if π/μ is an even integer.

In summary, we have derived the exact critical surface exponents

$$\alpha_s = 2 - \frac{\pi}{2\mu} \quad \text{and} \quad \alpha_1 = 1 - \frac{\pi}{\mu} \tag{23}$$

for the eight-vertex model. At $\mu = \pi/2$, $\alpha_s = 1$ (log) and $\alpha_1 = -1$ (log), in agreement with the Ising results [22–25]. Recalling the bulk exponents $\alpha_b = 2 - \pi/\mu$ [2] and $\nu = \pi/2\mu$ [5] we are thus able to provide a significant confirmation of the scaling relations [6,7,16] $\alpha_s = \alpha_b + \nu$ and $\alpha_1 = \alpha_b - 1$ between bulk and surface critical exponents. The derivation of other surface exponents awaits the diagonalization of the transfer matrix, which remains a formidable open problem.

We have found that the surface free energy scales as $f_s \sim p^{\pi/2\mu}$ as $p \to 0$. It is interesting to observe that this is in agreement with the scaling behavior of the interfacial tension [26]. However, these two quantities differ away from criticality.

Finally we note that, in the same spirit as this work, the critical magnetic surface exponent δ_s of the two-dimensional Ising model in a magnetic field [6,7] should also be obtainable from the elliptic dilute A_3 model [27], which is known to lie in the same universality class as the Ising model in a magnetic field [27]. The analog of the reflection equations for such face models have recently been written down by a number of authors (see [28] and [19]).

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