Floating of Extended States and Localization Transition in a Weak Magnetic Field

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We report results of a numerical study of noninteracting electrons moving in a random potential in two dimensions in the presence of a *weak* perpendicular magnetic field. We study the topological properties of the electronic eigenstates within a tight binding model. We find that in the weak magnetic field or strong randomness limit extended states float up in energy. Further, the localization length is found to diverge at the insulator phase boundary with the *same* exponent ν as that of the isolated lowest Landau band (high magnetic field limit).

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Recently there has been considerable interest in the fate of delocalized electronic states in a weak magnetic field in two dimensions (2D) [1–5]. In the limit of strong magnetic field, or equivalently weak randomness, it is believed that there exists a single critical energy within each Landau band where the localization length of electronic states diverges [6,7]. In contrast, one electron localization theory [8] predicts that in the absence of magnetic field all states are localized in 2D. Consequently, it was argued [9,10] that in the limit of weak magnetic field or strong randomness, where Landau bands merge together, these extended states do not disappear discontinuously but "float up," tending to infinite energy in the $B \to 0$ limit. Thus, for a given electron density (and hence finite Fermi energy E_F), for sufficiently low B all extended states are above E_F and the system becomes insulating. This scenario is crucial to the global phase diagram for the quantum Hall effect proposed by Kivelson *et al.* [11] and has received strong experimental support [1–3]. Recently, however, based on numerical calculations of localization length on a tight binding model (TBM), Liu, Xie, and Niu [5] concluded that extended states do not float. This issue is more clearly posed, and its resolution well described, by studying certain topological properties of the electronic eigenstates, as we shall see below.

A second issue of interest is the divergence of the localization length when approaching the insulator quantum Hall phase transition. A previous numerical study [12] performed on a random site TBM with a magnetic field suggested that the localization length exponent $\nu_i \simeq 0.8$ in 2D at the localization transition point. Besides the fact that this value is much smaller than that at the transition between quantum Hall phases in the strong magnetic field limit [7] $\nu_H \approx 2.4$, it violates the inequality $\nu \geq 2/d$ [13] which is widely believed to be satisfied in known random systems [14]. To address both these issues, a more clear-cut numerical method appears warranted.

In the presence of a magnetic field, electronic states exhibit interesting topological properties [15–18]. In particular, each state can be labeled by an integer called the Chern number, which is its boundary condition averaged Hall conductance, in units of e^2/h [16,17]. A state with nonzero Chern number carries Hall current and is necessarily extended. Thus by calculating the Chern numbers one is able to identify extended states unambiguously on *finite* size systems. This approach has proved very successful in addressing the localization problem in the lowest Landau band [19]. In this paper, we apply this approach to the TBM studied by Liu, Xie, and Niu [5] and also by Ando [20]. Our results clearly support the "floating up" picture and are consistent with Thouless number calculations by Ando [20]. In fact, the results of Liu, Xie, and Niu [5] are also consistent with ours, but our interpretation of their results is somewhat different, as we discuss later.

We also study the dependence of the number and energies of extended states on system size. We find, just as in the case of individual Landau bands, the localization length diverges only at individual energies. In the high field limit, the localization exponent is found to be the same as that of an isolated lowest Landau band, $\nu_H \approx 2.4$ [7]. For strong enough randomness the localization length remains finite throughout the band and the number of extended states goes to zero as the system size goes to infinity, in this one band model. Using finite size scaling, we find the largest localization length of the system diverges as the critical randomness is reached with an exponent ν_i which is the same as ν_H, contrary to previous suggestion [12,14] that they may be different. Thus our data show that ν is a universal exponent for all spin polarized integer quantum Hall transitions, including the ultimate one to the insulating state.

We study the TBM on a square lattice with nearest neighbor hopping, a uniform magnetic field, and random potential, described by the Hamiltonian

$$H = \sum_{mn}\{ - t(c^\dagger_{m+1,n}c_{m,n} + c^\dagger_{m,n+1}e^{i2\pi\alpha m}c_{m,n} + \mathrm{H.c.}) + \epsilon_{m,n}c^\dagger_{m,n}c_{m,n}\}, \quad (1)$$

where the integers m and n are the x and y coordinates of the lattice site, $c_{m,n}$ is the fermion operator on that site, t is the hopping matrix element which we set as the unit of energy from now on, and ϵ is the random

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potential ranging *uniformly* from $-W$ to W. α is the amount of magnetic flux per plaquette in units of the flux quantum hc/e. The Landau gauge $\mathbf{A} = (0, Bx, 0)$ is used in Eq. (1). Here we concentrate on the case $\alpha = 1/N_f$, where N_f is an integer. In this case, we have N_f Landau subbands in the absence of random potential, and the lowest energy subbands map onto the lowest Landau levels in the limit $N_f \to \infty$, which is the continuum limit.

The Hall conductance of an individual eigenstate $|m\rangle$ can be obtained easily using the Kubo formula [18]

$$\sigma_{xy}^m = \frac{ie^2\hbar}{A} \times \sum_{n\neq m} \frac{\langle m|v_y|n\rangle\langle n|v_x|m\rangle - \langle m|v_x|n\rangle\langle n|v_y|m\rangle}{(E_n - E_m)^2},$$

where A is the area of the system, and v_x and v_y are the velocity operators in the x and y directions, respectively. For a finite system with the geometry of a parallelogram with periodic boundary conditions (torus geometry), σ_{xy}^m depends on the two boundary condition phases ϕ_1 and ϕ_2. As shown by Niu *et al.*, the boundary condition averaged Hall conductance takes the form [16]

$$\langle\sigma_{xy}^m\rangle = \frac{1}{4\pi^2}\int d\phi_1\, d\phi_2\, \sigma_{xy}^m(\phi_1,\phi_2) = C(m)e^2/h\,, \tag{2}$$

where $C(m)$ is an integer called the Chern number of the state $|m\rangle$. States with nonzero Chern numbers carry Hall current and are necessarily extended states [17,19]. Thus by numerically diagonalizing the Hamiltonian on a grid of ϕ_1 and ϕ_2, and calculating the Chern numbers by converting the integral in (2) to a sum over grid points, we are able to identify extended states unambiguously.

We have studied systems of square geometry with various size (from 3×3 to 15×15), strength of randomness (W), and magnetic field (equivalently, N_f). The number of samples explored for a given W range from 2000 to 30 depending on system size. Most of our data were taken for $N_f = 3$. We do not, however, see any qualitative difference in behavior of the extended states, for systems with N_f as large as 13. Hence we believe our results are generic and apply to the continuum limit $N_f \to \infty$.

Figure 1 shows the density of states $[\rho(E)]$ and density of extended states with nonzero Chern numbers $[\rho_c(E)]$, for two different strengths of randomness for $N_f = 3$ on a square of lattice size 9×9. For weak enough randomness ($W = 1.0$), the three Landau subbands are broadened by randomness, but are still well separated. We see there are extended states in all subbands, with their densities peaked essentially at the center of each subband. This is consistent with the previous study on individual Landau bands [19]. As randomness increases, the subbands further broaden and start to merge, as is seen for $W = 2.5$. In this case there are still three prominent peaks in $\rho(E)$ (we call them E_1, E_2, and E_3, respectively), which are (loosely) identified as centers of Landau subbands. $\rho_c(E)$, however, now looks very different: most of the extended states are near the center of the entire band (E_2) and there is no peak

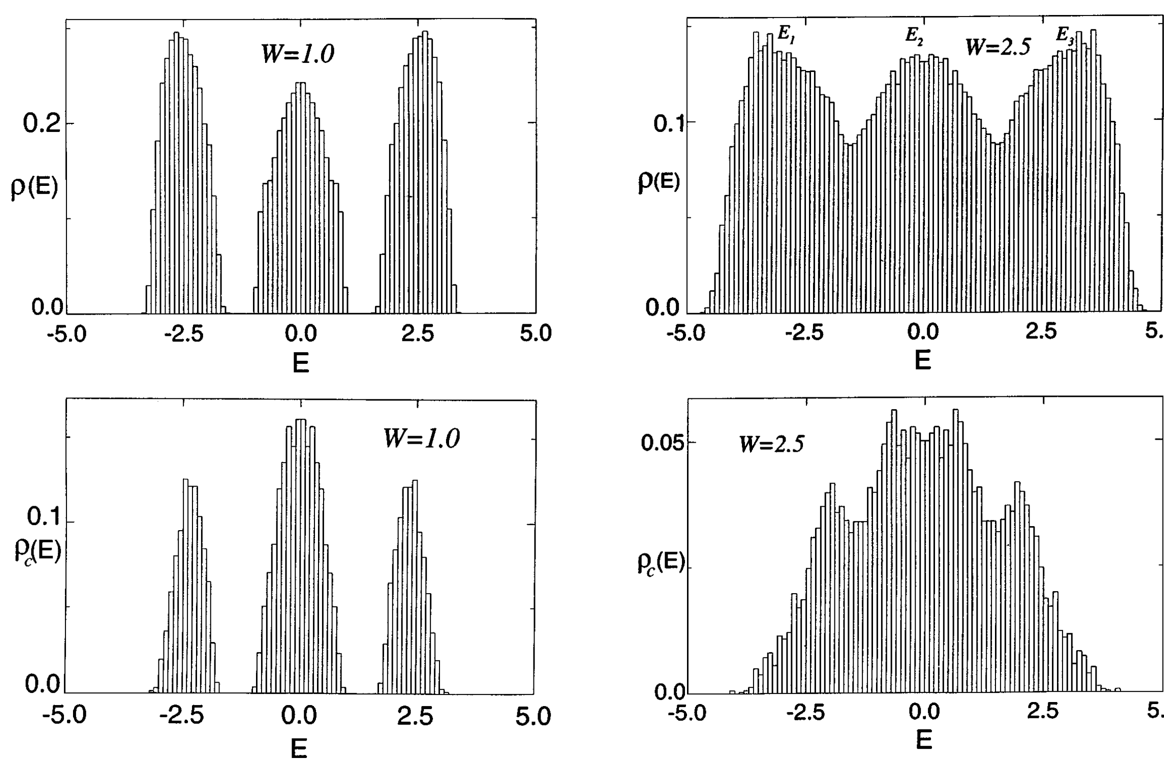

FIG. 1. Ensemble averaged density of states $\rho(E)$ and density of extended states $\rho_c(E)$ for two values of randomness W, for systems of size 9×9.

in $\rho_c(E)$ at E_1 or E_3, which are the centers of Landau subbands. There are nontrivial features in $\rho_c(E)$ which we discuss below, but it is clear from Fig. 1 that as the three subbands start to merge the extended states in the lower and upper subbands move away from the centers of the subbands (E_1 and E_3) toward center of the band (E_2). This behavior is also seen in systems of N_f as large as 13. We hence believe in the limit $N_f \to \infty$ (which can be mapped onto the continuum model) the extended states in the lowest subbands (which becomes Landau levels) float up toward the center of the band (which is at infinitely high energy relative to them in the continuum model). This provides unambiguous support for the floating up picture predicted theoretically [9,10] and seen experimentally [1–3].

The fact that the extended states in the lower and upper subbands float toward the center of the band as randomness increases may be understood in the following manner. In finite size systems, the Chern number of a state can change only when it becomes degenerate with a another state under certain boundary conditions. If such a degeneracy were to occur, the Chern numbers of the two states involved may change but their sum is conserved. Randomness tends to localize all states and annihilate the nonzero Chern numbers carried by the extended states. Thus states with nonzero Chern numbers of opposite signs "attract" each other and tend to move close in energy as randomness increases. It is believed that in the thermodynamic limit true extended states exist only at individual critical energies (see below). Each such critical energy is characterized by its total Chern number which is *invariant* as randomness varies, unless merging between critical energies occur. For exactly the same reason, critical energies with total Chern numbers of opposite sign also attract each other as randomness increases. In the case of $N_f = 3$ systems, the total Chern numbers for the three subbands are 1, -2, and 1, respectively [21]. Because of the "attraction," we expect that as randomness is turned on the extended states in the central subband with total Chern number -2 splits into two critical energies with total Chern number -1 each (by symmetry) and move toward the two band edges as randomness is increased further. Concurrently, the two critical energies of the upper and lower subbands with total Chern number $+1$ move away from the center of the subbands toward the center of the band. This is precisely what is seen in $\rho_c(E)$ at $W = 2.5$: There is a small dip at the center of the band indicating the splitting of the central critical energy; further, there are two less pronounced peaks from the two edge subbands, whose positions have clearly moved away from the corresponding peaks of $\rho(E)$. Similar behavior is found for systems with larger N_f. Our data suggest that the shift of critical energies from their corresponding peaks of $\rho(E)$ (floating) becomes sizable when the broadening of subbands due to randomness is comparable to subband gaps, consistent with previous prediction [9,10] that floating becomes important when $\omega_c\tau \sim 1$. The number of states below these critical energies (or, equivalently, the filling factor at these critical energies) grows roughly linearly with randomness strength as it further increases. However, we are unable to test the quantitative prediction of Refs. [9,10] due to system size limitations.

Figure 2 depicts the number of states with nonzero Chern number $N_c \equiv \int_{-\infty}^{\infty} \rho_c(E)\, dE$ versus the system size N_s (number of sites), for different values of disorder W, for $N_f = 3$, on a double logarithmic plot. We find the plot is essentially linear for small W up to $W \approx 3.0$, with slope $y = 0.79 \pm 0.01$ which is relatively independent of W, indicating that $N_c \sim (N_s)^y$ in this region. This power law behavior is exactly what is expected [18,19] where there are individual critical energies E_c^i in the vicinity of which the localization length diverges with a power law of the form $\xi(E) \sim |E - E_c|^{-\nu}$. In a finite system with linear size $L_s = \sqrt{N_s}$, states with $\xi(E) > L_s$ look extended. The number of such states goes like $N_c \sim N_s \rho(E_c) L_s^{-1/\nu} \sim N_s^{1-1/2\nu}$, thus $y = 1 - 1/2\nu$. This gives $\nu = 2.4 \pm 0.1$, in agreement with the ν_H for lowest Landau band [7,19]. This suggests that ν is a universal exponent in all spin-polarized integer quantum Hall transitions.

For larger W, the dependence of N_c on N_s deviates from a power law and bends down as N_s increases, indicating that the two critical energies have merged and disappeared; ξ is finite throughout the band. For strong enough randomness and large N_s, N_c *decreases* as N_s increases; thus in the localized regime the average number of extended states per sample goes to zero in the thermodynamic limit. From the shape of the density of extended states and scaling of data we determine the critical randomness to be $W_c \approx 2.9 \pm 0.1$. For W greater than but close to W_c, and large sizes N_s, N_c is expected to take the scaling form $N_c \sim N_s^y \tilde{F}(L_s/\xi_m) \sim N_s^y F(N_s^{1/(2\nu_i)}(W - W_c))$, where ξ_m is the largest localization length in the system that diverges as W_c is approached with exponent ν_i. The best

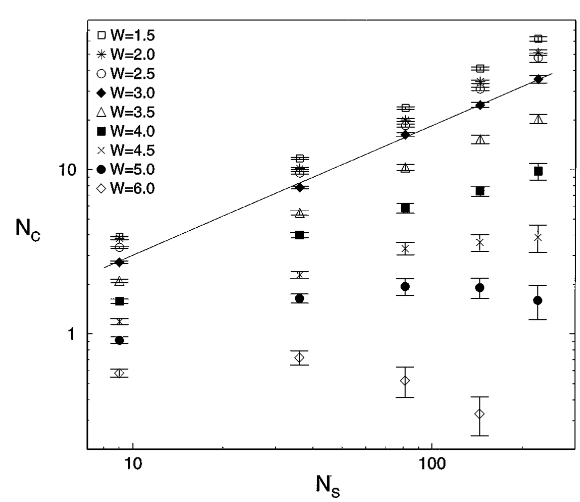

FIG. 2. Number of extended states N_c vs system size N_s for various W on a double logarithmic scale. The solid line with slope $y = 0.79$ is a linear fit to the data for $W = 3.0$.

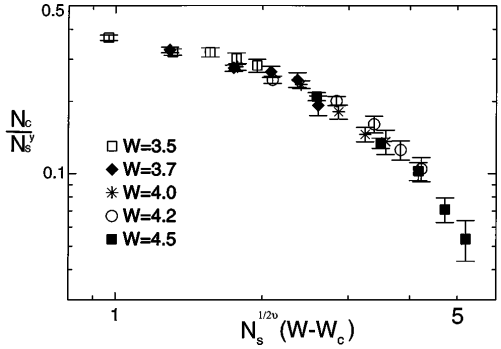

FIG. 3. The scaling function $F(N_s^{1/(2\nu_i)}(W - W_c))$.

scaling is achieved with $\nu_i \approx 2.3$, assuming $W_c = 2.9$, and Fig. 3 shows the scaling function F. Taking into uncertainty in W_c we estimate $\nu_i \approx 2.3 \pm 0.3$. This suggests that the localization length exponents are the same in both the localized and extended regimes, in contrast to a previous suggestion that they may be different [12]. The increasing negative slope of the scaling curve suggests that N_c goes to zero faster than any power law as N_s increases at large N_s.

We emphasize that the existence of this localized regime in the TBM is due to the fact that there exist critical energies with negative Chern numbers and the total Chern number of the system is zero. In the continuum, however, the total Chern number of the critical energy of each Landau band is 1, and there is *no* critical energy with negative Chern number at finite energy. Hence the extended states at these critical energies cannot annihilate their Chern numbers and become all localized as the randomness increases. They only float up and "disappear" at infinite energy. This becomes clear as one views the continuum system as the $N_f \rightarrow \infty$ limit of the TBM. In the TBM, the natural energy scale is hopping t (set to be 1 previously), and the zero point of energy is the center of the band. In the continuum, however, the energy scale is Landau level spacing $\hbar\omega_c$, and the zero point of energy is determined by identifying the center of the lowest energy band with energy $\hbar\omega_c/2$. In terms of TBM parameters we have $\hbar\omega_c = 4\pi t/N_f$. Based on our data up to $N_f = 13$ we conclude that the critical randomness is almost N_f independent and is about $W_c \approx 3t$, in agreement with Ando [20]. The energy at which the final merging and disappearance of critical energies measured from the *bottom* of the band is found to be of $O(W_c)$, which is the only energy scale of the TBM at criticality. Therefore the number of Landau subbands below the lowest critical energy (and hence its corresponding Landau level filling factor) before it finally disappears is of $O(W_c/\hbar\omega_c) \propto N_f$. We hence conclude in the continuum limit ($\hbar\omega_c$ finite, $N_f \rightarrow \infty$) the critical randomness strength ($W_c \propto N_f\hbar\omega_c$) is infinite, and extended states all float up to infinite energy in the strong randomness limit.

Liu, Xie, and Niu [5] find the energies of the extended states do not shift much relative to the *center* of the band ($E = 0$) and interpret it as evidence against floating. Our results for $N_f = 3$ are consistent with very little shift of peaks of $\rho_c(E)$ from edge bands relative to E_2. However, critical energies clearly float away from peaks of $\rho(E)$ (which are roughly at the centers of Landau subbands). This is because as randomness increases, both the bottom of the band and peaks in $\rho(E)$ move *downward*. We believe this *relative* movement is a clear indication of floating of extended states which survives the continuum limit.

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Note added.—After submitting this paper we became aware of a related work [22] where Chern numbers were calculated on the TBM with small sizes. We thank Q. Niu for bringing it to our attention.

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