Nonlinear Dynamics of a Driven Mode near Marginal Stability

H. L. Berk, B. N. Breizman,* and M. Pekker

Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712

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The nonlinear dynamics of a linearly unstable mode in a driven kinetic system is investigated to determine the saturated fields near the instability threshold. To leading order, this problem reduces to an integral equation with a temporally nonlocal cubic term. Its solution can exhibit self-similar behavior with a blowup in a finite time. When blowup occurs, the mode saturates due to plateau formation arising from particle trapping in the wave. Otherwise, the simplified equation gives a regular solution that leads to a saturation scaling reflecting the closeness to the instability threshold.

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In previous works [1–4], we have considered the nonlinear evolution of kinetic systems maintained by a balance of sources and relaxation processes that give rise to a distribution function with "free energy" [5,6] available to excite waves in a background medium such as a plasma. The instability mechanism is due to particles resonantly interacting with weakly unstable discrete modes, for which the linear growth rate γ is much less than the mode frequency ω. We assume that $\gamma = \gamma_L - \gamma_d$, where γ_L is the kinetic drive in the absence of dissipation, and γ_d is the intrinsic damping rate from the background plasma. Thus instability arises when $\gamma_L > \gamma_d$. In analysis given in past work it was assumed that $\gamma_d \ll \gamma_L$. The purpose of this Letter is to discuss the nonlinear character of this problem when $\gamma_d/\gamma_L \sim 1$, with particular emphasis given to the case near the instability threshold when $\gamma_L - \gamma_d \ll \gamma_L$.

In the presence of a wave of finite amplitude A resonant particles undergo nonlinear oscillations at a characteristic frequency ω_B that is proportional to $A^{1/2}$. These oscillations cause mode saturation due to the phase mixing of resonant particles that produces a local plateau in the resonance region of the distribution function as first discussed by Mazitov [7] and O'Neil [8]. For the specific one-dimensional bump-on-tail instability with $\gamma_d = 0$, it has been found that the maximum of ω_B is $3.2\gamma_L$ [9,10] for the initial value problem without particle sources and sinks.

We now consider the case $\gamma_d/\gamma_L \sim 1$ for the bump-on-tail problem. The results can be readily generalized to more general kinetic systems. We restrict the discussion to the case of isolated resonances, in which ω_B is less than the frequency separation between the resonances. For deeply trapped particles in the bump-on-tail problem, $\omega_B = (ek\hat{E}/m)^{1/2}$ where $\hat{E}\cos(\omega t - kx)$ is the perturbing longitudinal electric field. Using the particle simulation code described in Ref. [4], we have determined the maximum of the ratio $\omega_B/(\gamma_L - \gamma_d)$ as a function of γ_d/γ_L for the initial value problem and found that this ratio hardly changes as γ_d/γ_L is varied (the ratio varies from 3.2 to 2.9 as γ_d/γ_L varies from 0 to 0.6). This result implies that the particle distribution in the finite amplitude wave is only significantly altered from the unperturbed case in a region about the separatrix width. In this region the distribution "mixes," causing the formation of a plateau with the simultaneous conversion of the particle free energy into wave energy so that $\omega_B \sim \gamma$, which is the "natural" saturation level for pulsating cases which arise in the presence of a source and sufficiently strong background dissipation. However, even though $\omega_B \sim \gamma$ is a valid estimate, the simulations reveal an additional interesting feature. This can be observed in Figs. 1(a) and 1(b) which show the evolution of the wave amplitude in the initial value problem as a function of time for the cases $\gamma_d/\gamma_L = 0.05$ and 0.6, respectively. Note that the former case can be described by a predominantly single pulse of one sign with relatively small modulations during the decay phase of the pulse, an expected response. However, in the case $\gamma_d/\gamma_L = 0.6$ the amplitude versus time has deep modulations and reverses sign with the

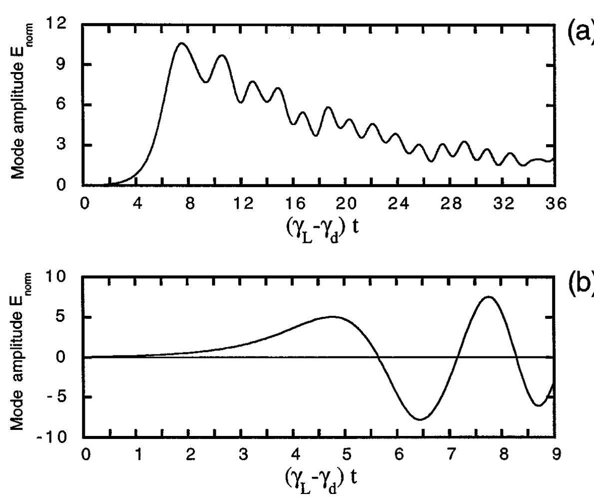

FIG. 1. Time evolution of the normalized wave amplitude $E_{\text{norm}} \equiv e\hat{E}k/m(\gamma_L - \gamma_d)^2$ in kinetic simulations of the bump-on-tail instability in the presence of background damping: (a) low-damping rate $\gamma_d/\gamma_L = 0.05$ and (b) damping rate comparable to the kinetic growth rate $\gamma_d/\gamma_L = 0.6$.

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highest maximum not immediately arising, a surprising result which we will discuss below.

When $\gamma \equiv \gamma_L - \gamma_d \ll \gamma_L$ one can expect to develop an analysis based on the closeness to marginal stability. For the sink that balances a constant source of particles we choose a particle annihilation model with ν the annihilation rate. We will assume $\nu \sim \gamma$ and that the relevant nonlinear time scale $\tau \sim 1/\gamma$ is shorter than ω_B^{-1}, the characteristic time it takes a trapped particle to complete a period. Hence we develop a perturbative analysis based on small deviations of the particles from their unperturbed orbits; formally we generate an expansion in the small parameter $(\omega_B \tau)^2$. Below we show that this procedure leads to the prediction of a steady state mode amplitude given by $\omega_B = 8^{1/4}\nu(\gamma/\gamma_L)^{1/4}$ which satisfies our assumption that $\omega_B\tau$ is small. This steady solution is only stable for $\nu > \nu_{\mathrm{cr}} \equiv 4.38\gamma$. For smaller ν values the amplitude is found to oscillate in time (close to the steady state one if $\nu_{\mathrm{cr}} - \nu \ll \nu_{\mathrm{cr}}$). However, when ν is sufficiently small, it is found from numerical integration and verified with a self-similar solution that the solution of the perturbatively derived equations blows up in a finite time. In reality this singular behavior leads to a level where the perturbation method fails. Saturation is then due to the natural saturation mechanism, where the distribution function flattens about the separatrix when ω_B rises to the level that it is $\sim\gamma$.

To begin the analysis we use a perturbative procedure to solve the equation for the distribution function $F(x, v, t)$ in the presence of an electric field, $E = \hat{E}(t)\cos(kx - \omega t + \alpha)$,

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} + \frac{e}{m}\hat{E}(t)\times \cos(kx - \omega t + \alpha)\frac{\partial F}{\partial v} + \nu F = S(v), \quad (1)$$

where e and m are the particle charge and mass, respectively, α is a phase which can be shown to remain constant in our problem, and $S(v)$ the source of particles. We will write F as a Fourier series

$$F = F_0 + f_0 + \sum_{n=1}^{\infty}[f_n \exp(in\psi) + \text{c.c.}], \quad (2)$$

where $F_0 = S(v)/\nu$ is the equilibrium distribution when $\hat{E} = 0$ and $\psi \equiv kx - \omega t + \alpha$.

The evolution equation for the wave amplitude is determined by the condition that the time rate of change of wave energy $\partial WE/\partial t$ is equal to the negative of the power dissipated into the background plasma $-2\gamma_d WE$ plus the power P the energetic particles transfer to the waves

$$P \doteq -\frac{e\omega}{k}\int dx\, dv\, E(x,t)F(x,v,t).$$

Note that for plasma waves the wave energy takes into account field energy and kinetic energy due to oscillations at the plasma frequency and is given by $WE = \int dx\, E^2(x,t)/4\pi$ where the x integration is over a wavelength. Now using these relations, we obtain

$$\frac{\partial \hat{E}(t)}{\partial t} = -\frac{4\pi e\omega}{k}\mathrm{Re}\int f_1\, dv - \gamma_d\hat{E}(t). \quad (3)$$

Thus we need to determine $\int dv\, f_1$ in terms of $\hat{E}(t)$ from Eq. (1) and substitute it into Eq. (3).

We assume that F can be expressed as a power series in $E(t)$ and we can truncate terms at sufficiently high n (we neglect $n \geq 3$). With $u = kv$, the equations for f_n ($n = 0, 1, 2$) are

$$\begin{aligned}
\frac{\partial f_0}{\partial t} + \nu f_0 &= -\frac{\omega_B^2}{2}\frac{\partial(f_1 + f_1^*)}{\partial u}, \\
\frac{\partial f_1}{\partial t} + iuf_1 + \nu f_1 &= -\frac{\omega_B^2}{2}\frac{\partial(F_0 + f_0 + f_2)}{\partial u}, \quad (4)\\
\frac{\partial f_2}{\partial t} + 2iuf_2 + \nu f_2 &= -\frac{\omega_B^2}{2}\frac{\partial f_1}{\partial u} + \mathcal{O}(\omega_B^2 f_3),
\end{aligned}$$

where $\omega_B^2 \equiv ek\hat{E}(t)/m$. These equations are integrated iteratively, assuming $F_0 \gg f_1 \gg f_2, f_0$ with the initial condition $F = F_0$. It turns out that f_2 does not contribute to the final result. By performing the time integration of Eqs. (4) we find $\int dv\, f_1(v,t)$ that reduces Eq. (3) to the form

$$\begin{aligned}
\frac{d}{dt}\omega_B^2 = (\gamma_L - \gamma_d)\omega_B^2(t) - \frac{\gamma_L}{2}\int_{t/2}^{t} dt'\,(t-t')^2\omega_B^2(t') \\
\times \int_{t-t'}^{t'} dt_1\, \exp[-\nu(2t - t' - t_1)] \\
\times\, \omega_B^2(t_1)\omega_B^2(t' + t_1 - t), \quad (5)
\end{aligned}$$

where $\gamma_L = 2\pi^2(e^2\omega/mk^2)\partial F_0(\omega/k)/\partial v$. We rescale our variables with the transformations $\tau = (\gamma_L - \gamma_d)t$, $A = [\omega_B^2/(\gamma_L - \gamma_d)^2][\gamma_L/(\gamma_L - \gamma_d)]^{1/2}$, $\hat{\nu} = \nu/(\gamma_L - \gamma_d)$. Equation (5) can then be written as

$$\begin{aligned}
\frac{dA}{d\tau} = A(\tau) - \frac{1}{2}\int_0^{\tau/2} dz\, z^2 A(\tau - z) \\
\times \int_0^{\tau - 2z} dx\, \exp[-\hat{\nu}(2z + x)] \\
\times A(\tau - z - x)A(\tau - 2z - x). \quad (6)
\end{aligned}$$

Note that $\hat{\nu}$ is the only parameter appearing in Eq. (6).

As long as the solution to Eq. (6) remains finite, the amplitude A for $\hat{\nu} \ll 1$ will be a dimensionless and scale-free number, which implies that $\omega_B/(\gamma_L - \gamma_d) \sim (1 - \gamma_d/\gamma_L)^{1/4}$, which is smaller than the natural saturation level if $1 - \gamma_d/\gamma_L \ll 1$.

We find that Eq. (6) admits a constant solution A_0 as $\tau \to \infty$,

$$A_0 = 2\sqrt{2}\,\hat{\nu}^2. \tag{7}$$

We examine the stability of this solution by looking for solutions of the form

$$A(t) = A_0 + \delta A e^{\hat{\nu}\lambda\tau}, \tag{8}$$

where $\hat{\nu}\lambda$ is the eigenvalue and instability arises if $\mathrm{Re}\lambda > 0$. Substituting Eq. (8) into Eq. (5) leads to the dispersion relation

$$\hat{\nu} = \frac{1}{\lambda}\left[1 - \frac{8}{(1+\lambda)(2+\lambda)^2} - \frac{1}{(1+\lambda)^4}\right]. \tag{9}$$

Instability is found to arise when $\hat{\nu} < \hat{\nu}_{\rm cr} \equiv 4.38$ (for $\hat{\nu} = \hat{\nu}_{\rm cr}$, we find $\lambda = \pm 0.46i$).

In Fig. 2 we show numerical solutions of Eq. (6) for various values of $\hat{\nu}$ starting with sufficiently small values of A so that initially the nonlinear term in Eq. (6) is unimportant. In Fig. 2(a), with $\hat{\nu} = 5$ we see that $A(t)$ goes to the steady state value $2\sqrt{2}\,\hat{\nu}^2$. With $\hat{\nu} = 4.3$, we see in Fig. 2(b) that the solution pulsates periodically in time around the steady state level; analytically one finds for $\nu_{\rm cr} - \nu \ll \nu_{\rm cr}$ that

$$A(t) = 2\sqrt{2}\,\hat{\nu}^2(1 + \mu(\hat{\nu}_{\rm cr} - \hat{\nu})^{1/2} \times \cos\{[2.01 + \beta(\hat{\nu}_{\rm cr} - \hat{\nu})]t\})$$

with $\mu = 0.76$, $\beta = 0.8$. For $\hat{\nu} = 3$, we see in Fig. 2(c) that the oscillation amplitude exceeds the steady level so that $A(t)$ even changes sign. In Fig. 2(d), we show results of the $\hat{\nu} = 2.5$ case where the oscillations have become irregular, indicating bifurcations to other periods have taken place. In Fig. 2(e), for $\hat{\nu} = 2.4$, we see that the system breaks into oscillations with decreasing periods and with ever increasing amplitude.

This final behavior is predicted by a self-similar singular solution of Eq. (6) that blows up in a finite time. Such a solution needs to have an oscillatory structure for $A(t)$ as without oscillations, it is readily demonstrated that the cubic term will stabilize the linear terms [e.g., this occurs in obtaining the steady solution given by Eq. (7)]. As the blowup occurs very quickly, $\dot{A} \gg A$, so that the first term on the right-hand side of Eq. (6) is unimportant. We then

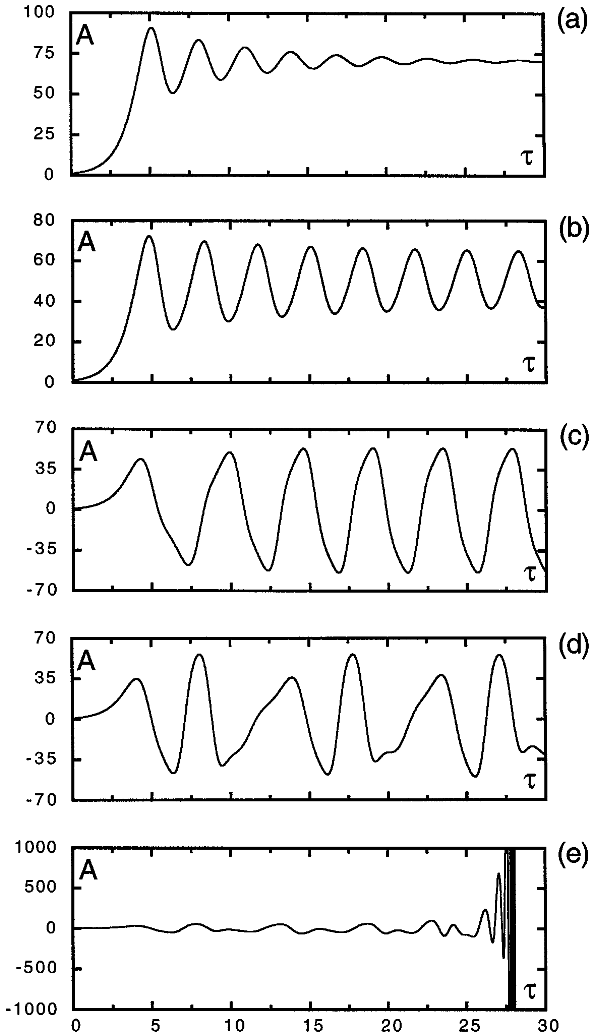

FIG. 2. Numerical solutions of Eq. (6) for $A(0) = 1$ and various values of $\hat{\nu}$: (a) $\hat{\nu} = 5.0$, (b) $\hat{\nu} = 4.3$, (c) $\hat{\nu} = 3.0$, (d) $\hat{\nu} = 2.5$, and (e) $\hat{\nu} = 2.4$.

seek a solution of the form

$$A(t) = g(\alpha \ln(t_0 - t))/(t_0 - t)^p, \tag{10}$$

assuming $\hat{\nu}(t_0 - t) \ll 1$ and where $g(\chi)$ is a periodic function of χ. The choice of this form enables us to have a balance between $\dot{A}$ and the nonlinear term in Eq. (6). In particular, observe that $(t_0 - t)^{-(p+1)}$ factors from the quantity

$$\dot{A}(t) = \frac{1}{(t_0 - t)^{p+1}}\left[pg - \alpha\frac{\partial g(\chi)}{\partial \chi}\right].$$

The choice $p = 5/2$ allows $(t_0 - t)^{-(p+1)}$ to be factored from the nonlinear term as well. Thus the problem is reduced to finding a periodic function $g(\chi)$. We take $\hat{\nu}(t_0 - t) \ll 1$ and use the fact that the expected solution diverges near $t = t_0$. This allows us to extend the upper integration limits of Eq. (6) to infinity, giving

$$\frac{5}{2} g - \alpha \frac{\partial g}{\partial \chi} = -\frac{1}{2} \int_0^\infty d\xi\, g(\chi + \alpha \ln(1+\xi)) \times \int_0^\infty d\eta\, V(\xi;\eta) g(\chi + \alpha \ln(1+\xi+\eta)) g(\chi + \alpha \ln(1+2\xi+\eta)), \qquad (11)$$

where

$$V(\xi;\eta) \equiv \xi^2/(1+\xi)^{5/2}(1+\xi+\eta)^{5/2}(1+2\xi+\eta)^{5/2}.$$

We look for a Fourier solution for $g(\chi)$ of the form $g(\chi) = \frac{1}{2}\sum_{n=0}^{\infty}(g_{2n+1}e^{i(2n+1)\chi} + \text{c.c.})$, and we attempt to solve this equation by iteration in g_{2n+1}. If we first neglect g_{2n+1} for $n \geq 1$, we find that α satisfies the equation

$$-\frac{2\alpha}{5} = \frac{\int_0^\infty d\xi \int_0^\infty d\eta\, V(\xi;\eta)[\sin(\ln a_1^\alpha) + \sin(\ln a_2^\alpha) + \sin\ln(a_3^\alpha)]}{\int_0^\infty d\xi \int_0^\infty d\eta\, V(\xi;\eta)[\cos(\ln a_1^\alpha) + \cos(\ln a_1^\alpha) + \cos\ln(a_3^\alpha)]}, \qquad (12)$$

where

$$a_1 = \frac{(1+\xi)(1+\xi+\eta)}{1+2\xi+\eta},$$

$$a_2 = \frac{(1+\xi)(1+2\xi+\eta)}{1+\xi+\eta},$$

$$a_3 = \frac{(1+\xi+\eta)(1+2\xi+\eta)}{1+\xi}.$$

Equation (12) admits the solution $\alpha = 11.67$. If the iteration is carried out to the next order, the correction to α is less than 0.01, which indicates that the iteration procedure leads to a rapidly convergent series. Note that the above solution is not unique. We have found that Eq. (11) also has another solution that contains both odd and even Fourier components, so that

$$g(\chi) = \frac{1}{2}\sum_{n=0}^{\infty}(g_n e^{in\chi} + \text{c.c.}).$$

For this solution, α is close to 6.1. Depending on initial conditions, the system may asymptote to either solution. In our numerical simulations we find that the $(t_0 - t)^{-5/2}$ divergence is robust, and the oscillatory behavior is fitted relatively well with $\alpha \doteqdot 6.1$.

We also observed that even for $\hat{\nu} > \hat{\nu}_{\text{cr}}$ we can find a diverging solution of Eq. (6) if the initial amplitude is large enough.

In summary, we have completed the understanding of the wave saturation mechanisms of isolated weakly unstable modes in kinetic systems destabilized by resonant particles. The new element in this work is the quantitative description of the dynamics near instability threshold. New scaling features have been found for both steady state and pulsating solutions. Surprisingly, we find that the system with a sufficiently weak source reaches the saturation levels that are expected from particle trapping, $\omega_B \sim \gamma_L - \gamma_d$, even though the dimensionless scaling of the equation would indicate that the saturation level should scale as $\omega_B \sim (\gamma_L - \gamma_d)(1 - \gamma_d/\gamma_L)^{1/4}$.

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*Also at Budker Institute of Nuclear Physics, Novosibirsk 630090, Russia.

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