Estimating Model Parameters from Time Series by Autosynchronization

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Parameters of a given model describing a (chaotic) dynamical system are estimated from scalar time series using autosynchronization where the parameter adaption process is controlled by the synchronization of the model to the given dynamics. A practical method is presented for deriving the necessary ordinary differential equations for the parameter controlling loop.

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The synchronization of (unidirectionally) coupled dynamical systems and its possible applications in communication schemes is currently a field of great interest (see [1–7] and references cited therein). In this Letter we discuss a special feature of synchronizing systems called *autosynchronization* where a system with slowly varying parameters converges from a state of nonsynchronization to synchronization. This adaption process is governed by additional ordinary differential equations (ODEs) for the parameters that are controlled by the synchronization error. A systematic way for deriving the parameter controlling loop is presented and illustrated by numerical examples. For the sake of brevity we consider unidirectionally coupled systems only, although the main ideas can in principle also be applied to mutually coupled synchronizing systems. In order to indicate a possible application in nonlinear time series analysis [8] and system identification autosynchronization is discussed and used in the following for estimating the parameters of a given model from a scalar time series [9–14].

Let

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}) \tag{1}$$

be the (experimental) dynamical system whose parameters $\mathbf{p} \in R^m$ are to be estimated. The only information available is a time series $s(t)$ given by a (scalar) observable

$$s = h(\mathbf{x}) \tag{2}$$

and the structure of the model $\mathbf{f}$. Furthermore, let us assume that we are able to construct a dynamical system

$$\dot{\mathbf{y}} = \mathbf{g}(s, \mathbf{y}, \mathbf{q}) \tag{3}$$

that synchronizes ($\mathbf{y} \rightarrow \mathbf{x}$ for $t \rightarrow \infty$) if $\mathbf{q} = \mathbf{p}$. If the functional form of the vector field $\mathbf{f}$ is known, such a system can, for example, be constructed by the subsystem decomposition introduced by Pecora and Carroll [3] or, more generally, by an active-passive decomposition of $\mathbf{f}$ [5,6]. The unidirectionally coupled systems (1) and (3) are called *drive* and *response*, respectively. The main question addressed in this Letter is: "Can we find a set of ODEs for the parameters $\mathbf{q}$ of system (3)

$$\dot{\mathbf{q}} = \mathbf{u}(s, \mathbf{y}, \mathbf{q}) \tag{4}$$

such that $(\mathbf{y}, \mathbf{q}) \rightarrow (\mathbf{x}, \mathbf{p})$ for $t \rightarrow \infty$ and is there a practical and systematical way to derive it?"

The answer to both parts of this question is "yes." In order to see that it is in principle possible to find such an additional controlling loop for the (unknown) parameters $\mathbf{q}$ consider the following example that is based on the well-known Lorenz system [15]:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= p_1 x_1 - p_2 x_2 - x_1 x_3 + p_3, \\ \dot{x}_3 &= x_1 x_2 - b x_3, \end{aligned} \tag{5}$$

with $p_1 = r = 28$, $p_2 = 1$, $p_3 = 0$, $\sigma = 10$, and $b = 8/3$. We assume that the time series available is given by the observable

$$s = h(\mathbf{x}) = x_2. \tag{6}$$

The model to be fitted to the data is driven by s and is written

$$\begin{aligned} \dot{y}_1 &= \sigma(s - y_1), \\ \dot{y}_2 &= q_1 y_1 - q_2 y_2 - y_1 y_3 + q_3, \\ \dot{y}_3 &= y_1 y_2 - b y_3. \end{aligned} \tag{7}$$

Using a global Lyapunov function one can show that for $\mathbf{q} = \mathbf{p}$ synchronization ($\mathbf{y} \rightarrow \mathbf{x}$) occurs for all initial conditions [6]. As ODEs for the parameter controlling loop we use

$$\begin{aligned} \dot{q}_1 &= u_1(s, \mathbf{y}, \mathbf{q}) = [s - h(\mathbf{y})] y_1 = (x_2 - y_2) y_1, \\ \dot{q}_2 &= u_2(s, \mathbf{y}, \mathbf{q}) = [s - h(\mathbf{y})] y_2 = -(x_2 - y_2) y_2, \\ \dot{q}_3 &= u_3(s, \mathbf{y}, \mathbf{q}) = s - h(\mathbf{y}) = x_2 - y_2. \end{aligned} \tag{8}$$

To prove that $(\mathbf{y}, \mathbf{q}) = (\mathbf{x}, \mathbf{p})$ is a globally stable solution of the response systems (7) and (8) we investigate the dynamics of the differences $\mathbf{e} = \mathbf{y} - \mathbf{x}$ and $\mathbf{f} = \mathbf{q} - \mathbf{p}$ which is given by the following set of differential equations

$$\begin{aligned} \dot{e}_1 &= -\sigma e_1, \\ \dot{e}_2 &= q_1 y_1 - p_1 x_1 - q_2 y_2 + p_2 x_2 - y_1 y_3 + x_1 x_3 + f_3, \\ \dot{e}_3 &= y_1 y_2 - x_1 x_2 - b e_3, \\ \dot{f}_1 &= -e_2 y_1, \\ \dot{f}_2 &= e_2 y_2, \\ \dot{f}_3 &= -e_2, \end{aligned} \tag{9}$$

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where the parameters $\mathbf{p}$ have been assumed to be constant ($\dot{\mathbf{p}} = \mathbf{0}$). The first equation in (9) implies $e_1 \rightarrow 0$, i.e., $y_1 \rightarrow x_1$. For the limit $t \rightarrow \infty$ the remaining equations can therefore be written as

$$\begin{aligned}
\dot{e}_2 &= y_1 f_1 - y_2 f_2 - p_2 e_2 - y_1 e_3 + f_3\,,\\
\dot{e}_3 &= y_1 e_2 - b e_3\,,\\
\dot{f}_1 &= -e_2 y_1\,,\\
\dot{f}_2 &= e_2 y_2\,,\\
\dot{f}_3 &= -e_2\,,
\end{aligned} \tag{10}$$

Since the derivative of the Lyapunov function $L = e_2^2 + e_3^2 + f_1^2 + f_2^2 + f_3^2$ is for positive values of the parameter p_2 strictly negative, $\frac{1}{2}\dot{L} = -p_2 e_2^2 - b e_3^2$, the response systems (7) and (8) converges globally to the parameters $\mathbf{p}$ of the original system (5) and synchronizes. This autosynchronization is illustrated in Fig. 1(a) for the initial conditions $\mathbf{x} = (0.1, 0.1, 0.1)$, $\mathbf{y} = (-0.1, 0.1, 0)$, and $\mathbf{q} = (10, 10, 10)$. For better visualization the first parameter q_1 has been divided by 10 and the dotted lines give the exact values $p_1/10 = 2.8$, $p_2 = 1$, and $p_3 = 0$. In this case we have assumed that the other parameters of the drive and the response system coincide exactly. To demonstrate the influence of discrepancies of the parameters that are not recovered but kept fixed Fig. 1(b) shows an example where the parameter $\sigma = 10$ of the drive system (5) was replaced by $\sigma = 10.1$ in the response system (8). In this case the parameters $\mathbf{q}$ converge not exactly but oscillate near the true values $\mathbf{p}$. Note that the parameter q_2 is most sensitive to the parameter mismatch whereas q_1 remains very close to p_1.

This example shows that it is in principle possible to recover several parameters of a model from a time series using autosynchronization. In general, however, an analytical treatment of the problem is not possible and a practical approach for deriving the differential equations of the parameters is desirable. Such a method will now be presented. For the purpose of motivation we consider again systems (5)–(8). To simplify graphical illustrations, however, we assume that only p_1 and p_2 are to be recovered.

The dynamics of the parameters q_1 and q_2 is governed by the vector fields u_1 and u_2 given in Eq. (8). For *constant* values of the parameters q_1 and q_2 we can compute the average controlling forces U_k ($k = 1, 2$)

$$U_k = \lim_{T\rightarrow\infty} \frac{1}{T} \int_0^T u_k\, dt\,. \tag{11}$$

In Fig. 2 the values of these forces are plotted as functions of q_1 and q_2. For $(q_1, q_2) = (p_1, p_2)$ the functions U_1 and U_2 vanish.

Our goal now is to derive (optimal) differential equations for the parameters $\mathbf{q} = (q_1, q_2)$ using as a starting point the functions $U_k(q_1, q_2)$ that can be computed numerically for any given model. For this purpose we first compute the gradients $\mathbf{g^k}$ of U_k at $(q_1, q_2) = (p_1, p_2)$. This can be done by fitting locally a linear map of the form $U_k \approx \langle \mathbf{g^k}, \mathbf{q} - \mathbf{p} \rangle$ to the numerically computed values of U_k near $\mathbf{q} = \mathbf{p}$. In Figs. 2(c) and 2(d) the directions of the gradients are denoted by arrows.

Along these gradients the desired action of the parameter controlling loop is easy to formulate: parameter values $\mathbf{q}$ have to be shifted in parallel to the gradient until they reach the borderline between the hatched and the blank region where $U_k = 0$. This controlling strategy can be

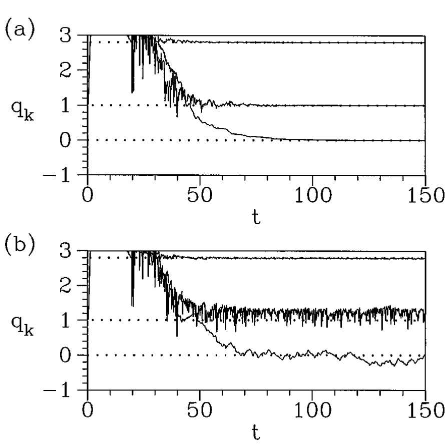

FIG. 1. Convergence of the recovered parameter values $q_1/10$, q_2, and q_3 of the response system (7) and (8) to the fixed values $p_1/10 = 2.8$, $p_2 = 1$, and $p_3 = 0$ of the drive (5). (a) All parameters except for $\mathbf{q}$ coincide exactly. (b) The parameter σ equals 10 in the drive and 10.1 in the response.

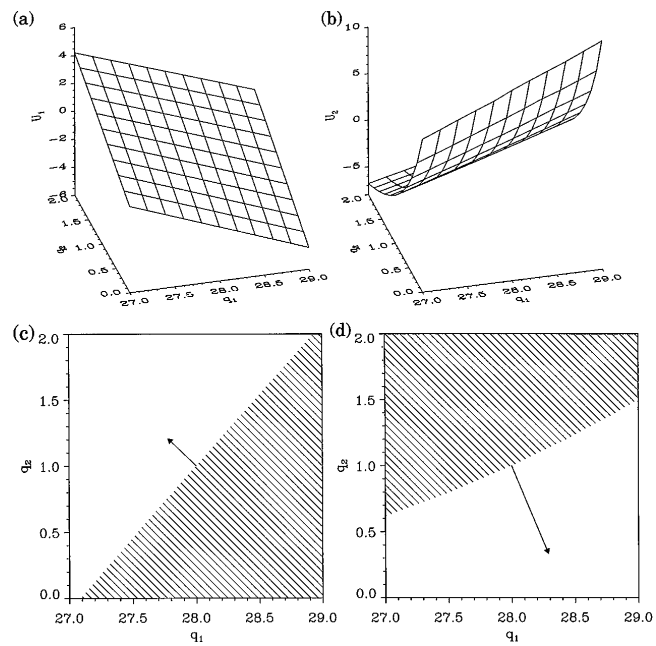

FIG. 2. Averaged controlling forces U_1 and U_2 [Eq. (11)] plotted vs q_1 and q_2. (a),(b) Surface plots. (c),(d) Contour plots. In the hatched regions the functions U_k are negative. The arrows denote the directions of the gradients $\mathbf{g^k}$ ($k = 1, 2$).

implemented most easily if the gradients coincide with the axes of the parameter space. Therefore we change the parameter coordinate system by projecting the parameter vector $\mathbf{q}$ onto the gradients $\mathbf{g^k} = (g_1^k, g_2^k)$

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = A\mathbf{q} . \tag{12}$$

In the new r-coordinate system of the parameter space the dynamics for the desired parameter correction may be written as

$$\dot{r}_k = -U_k . \tag{13}$$

Expressed in the original q-coordinate system the controlling equations are given by

$$\dot{\mathbf{q}} = A^{-1}\dot{\mathbf{r}} = -A^{-1}\mathbf{U} . \tag{14}$$

If the parameter variations are much slower than the time scale of the (chaotic) dynamics, the temporal averages U_k can be replaced by the functions u_k and we obtain

$$\dot{\mathbf{q}} = \alpha B \mathbf{u} , \tag{15}$$

where $B = -A^{-1}$ and α is a free parameter that has been added to control the speed of convergence. In our example the gradients are given by $\mathbf{g^1} = (-2.24, 2.08)$ and $\mathbf{g^2} = (2.91, -6.70)$ and the resulting ODEs for the parameters q_1 and q_2 are

$$\begin{aligned} \dot{q}_1 &= 0.748(s - y_2)y_1 - 0.232(s - y_2)y_2 , \\ \dot{q}_2 &= 0.325(s - y_2)y_1 - 0.250(s - y_2)y_2 , \end{aligned} \tag{16}$$

where we used $\alpha = 1$. Figure 3 shows a comparison of the convergence properties of q_1 and q_2 in the case that the parameter σ of the response is 1% larger than the value $\sigma = 10$ of the drive [compare Fig. 1(b)]. For the results shown in Fig. 3(a) the first two ODEs of Eqs. (8) were used. The computations illustrated in Fig. 3(b) are based on the derived parameter ODEs given in Eq. (16). As can be seen in particular the convergence to $p_2 = 1$ is less erratic for the derived parameter controlling loop. When comparing Eq. (16) with Eq. (8) one can see that the ODE for q_2 differs more than that for q_1. Since the derived ODEs Eq. (16) are in some sense optimal, this gives some explanation for the different convergence properties of q_2 observed in Fig. 3.

The crucial point of this approach is the proper selection of the functions u_k defining the controlling forces. Of course, these functions have to vanish for $(\mathbf{y}, \mathbf{q}) = (\mathbf{x}, \mathbf{p})$. This can be achieved by a product ansatz $u_k = [s - h(\mathbf{y})] \cdot \tilde{u}_k$ with $\tilde{u}_k = \tilde{u}_k(s, \mathbf{y}, \mathbf{q})$. Furthermore, the averaged controlling forces U_k should be smooth functions of the parameters $\mathbf{q}$ near $\mathbf{p}$ changing their signs along a (smooth) curve passing through $\mathbf{p}$ (compare Fig. 2).

In the last example we have used this strategy to establish parameter ODEs for the three original parameters $\mathbf{q} = (\sigma, r, b)$ of the Lorenz model. The resulting equations are

$$\begin{aligned} \dot{q}_1 &= 0.07(-0.786u_1 + 11.2u_2 + 59.2u_3) , \\ \dot{q}_2 &= 0.07(4.05u_1 - 87.0u_2 - 261u_3) , \\ \dot{q}_3 &= 0.07(-0.518u_1 + 12.3u_2 + 36.4u_3) , \end{aligned} \tag{17}$$

with

$$\begin{aligned} u_1 &= (s - y_2)y_2 , \\ u_2 &= (s - y_2)\frac{y_1}{10 + y_2^2} , \\ u_3 &= (s - y_2)\frac{y_2}{10 + y_1^2} . \end{aligned} \tag{18}$$

Figure 4 shows the convergence of the three parameters $q_1/10 = \sigma/10$, $q_2/10 = r/10$, and $q_3 = b$, where again q_1 and q_2 are rescaled by 10 for better visualization.

In this Letter we have shown that it is possible to design dynamical systems that are able to adapt to a given time series by parameter variations that are controlled by the synchronization error of the system with respect to the

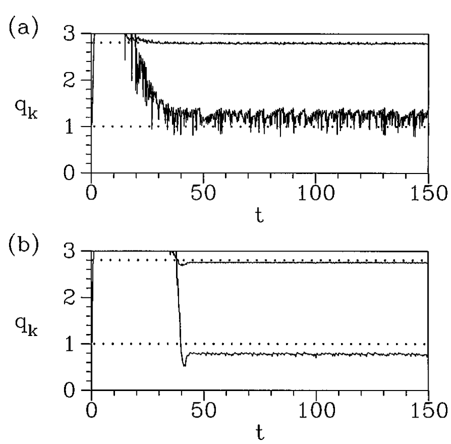

FIG. 3. Convergence of the recovered parameters $q_1/10$ and q_2 of the response system (7) and (8) to the fixed values $p_1/10 = 2.8$ and $p_2 = 1$ of the drive (5). (a) Using the parameter ODEs (8). (b) Using the derived parameter ODEs (16).

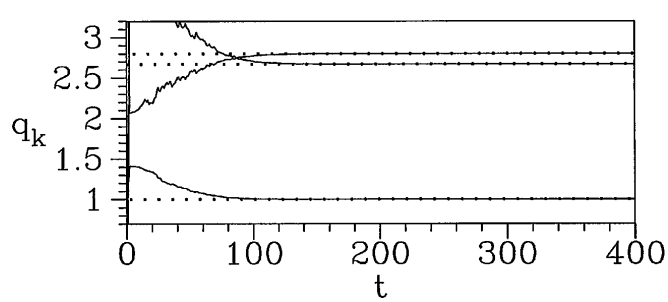

FIG. 4. Convergence of the recovered Lorenz parameters to the true values $p_1/10 = \sigma/10 = 1$, $p_2/10 = r/10 = 2.8$, and $p_3 = b = 8/3$. For the parameter controlling loop the ODEs (17) have been used.

time series. The method has also been tested for its robustness with respect to additive noise simulating the case that a noisy time series has been measured. As long as the synchronization is not completely destroyed the recovered parameters fluctuate around their true values. The amplitude of these fluctuations can be reduced by decreasing the convergence parameter α in Eq. (15), which of course also reduces the speed of convergence. An important task for future research is a systematic comparison with other methods for parameter estimation [9–14]. An advantage of the method presented here is the fact that once the parameter ODEs have been derived the computation of the parameters requires no CPU time or storage intensive arithmetics and may even be implemented as a realtime algorithm and/or using analog computers or electronic circuits [16]. Furthermore, the autosynchronization process cannot only be used for estimating the parameters of a given model from a time series but offers various potential applications. For example, the parameters of the drive may be modulated slowly in order to encode a message [4–6] that consists of different information signals. At the response system the corresponding parameters follow this modulation and in this way several information signals can be decoded from a *single* chaotic signal that is transmitted from the drive (transmitter) to the response (receiver) [16]. Other possible applications include monitoring of technical devices [17] or the development of measurement techniques based on sensitive dependence on parameters.

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