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Diffeomorphism Groups, Anyon Fields, and q Commutators

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We construct explicit anyon fields from unitary representations of the group diffeomorphisms of the plane, realizing braid group elements as paths in the plane transforming naturally under diffeomorphisms. The fields satisfy q-commutation relations, where q is the anyonic phase shift.

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The first rigorous prediction of anyon statistics, confirming a conjecture of Leinaas and Myrheim [1], came from interpreting certain representations of the group of diffeomorphisms of the plane [2]. This led to many fundamental physical properties of anyons, and to the role of the braid group [3]. Anyon statistics find application to surface phenomena, particularly the fractional quantum Hall effect [4–6]. In this Letter we construct creation and annihilation fields $\psi^*(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$ as operators *intertwining a hierarchy* of N-anyon diffeomorphism group representations. These fields obey q-commutation relations; i.e., the q commutator becomes the fundamental bracket of anyon field theory. This bracket is not a starting assumption [7] or the result of introducing a Chern-Simons potential into a canonical theory, nor do we obtain it by q-deforming Bose or Fermi quantum mechanics. Surprisingly, it is strictly a consequence of the group representations describing anyons, together with the (completely general) intertwining property of the fields. The latter property is motivated geometrically and entails commutator brackets only. Our development includes an interesting way to realize the braid group, with diffeomorphisms of $\mathbf{R}^2$ acting on its elements.

First we state some basic facts about the Lie algebra of mass and momentum density operators $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$, the corresponding Lie group, and its unitary representations, and propose the general formula satisfied by intertwining fields. Fixing t, the spatially averaged operators $\rho(f) = \int \rho(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}$ and $J(\mathbf{g}) = \int \mathbf{J}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) d\mathbf{x}$ (for smooth functions f and vector fields $\mathbf{g}$) generate an infinite-dimensional, nonrelativistic local current algebra,

$$[\rho(f_1), \rho(f_2)] = 0, \qquad [\rho(f), J(\mathbf{g})] = i\hbar\rho(\mathbf{g}\cdot\nabla f),$$
$$[J(\mathbf{g}_1), J(\mathbf{g}_2)] = -i\hbar J([\mathbf{g}_1, \mathbf{g}_2]), \tag{1}$$

where $[\mathbf{g}_1, \mathbf{g}_2] = \mathbf{g}_1 \cdot \nabla \mathbf{g}_2 - \mathbf{g}_2 \cdot \nabla \mathbf{g}_1$ is the usual Lie bracket. Now each $\mathbf{g}$ generates a *flow*, i.e., a one-parameter group of diffeomorphisms $\phi_s^{\mathbf{g}}$ ($s \in \mathbf{R}$). The unitary operators $U(f) = \exp[(i/m)\rho(f)]$ and $V(\phi_s^{\mathbf{g}}) = \exp[(is/\hbar)J(\mathbf{g})]$ represent a semidirect product group,

$$U(f_1)V(\phi_1)U(f_2)V(\phi_2) = U(f_1 + f_2 \circ \phi_1)V(\phi_1\phi_2), \tag{2}$$

where $\phi_1\phi_2$ is the composition of diffeomorphisms.

In general, the manifold $\mathbf{M}$ where (2) applies is the *physical space* of the theory. Usually $\mathbf{M} = \mathbf{R}^3$, while anyons occur when $\mathbf{M}$ is two dimensional. For fixed $\mathbf{M}$, inequivalent representations of (1) and (2) describe distinct systems. This perspective, established in our earlier work, leads to a unified description of an astonishing variety of quantum theories: point particles obeying Bose, Fermi, or fractional statistics, infinite systems in the thermodynamic limit, and extended objects such as vortex configurations [2,3,8]. Here canonical fields ψ and ψ^* do not play a fundamental role; particle statistics, formerly described by the field algebra, is now described by the group representation. The case has been made that (2) defines a universal, or generic, group of local symmetries for nonrelativistic quantum theory.

Nevertheless, it is natural to ask how such fields could be constructed, given a set of representations of (1) and (2). This question is answered next. The simplest unitary representations of (2) are the N-particle representations. The Bose (Fermi) representations form a *hierarchy* in an obvious physical sense that we now make precise. Let $U_N(f)$ and $V_N(\phi)$ be unitary representations of (2) in $\mathcal{H}_N$, describing systems of N identical configurations. Let $h \in \mathcal{H}_1$, and let $\psi^*(h), \psi(h)$ be intertwining operators labeled by h; $\psi^*(h) : \mathcal{H}_N \to \mathcal{H}_{N+1}$ and $\psi(h) : \mathcal{H}_{N+1} \to \mathcal{H}_N$, with $\psi(h)$ annihilating the vacuum state $\Omega_0 \in \mathcal{H}_0$. Thus $\mathcal{H}_1$ establishes the nature of the configuration that ψ^* creates and ψ annihilates, while h describes its state. We now propose the conditions

$$\begin{aligned} U_{N+1}(f)\psi^*(h) &= \psi^*(U_{N=1}(f)h)U_N(f)\,, \\ V_{N+1}(\phi)\psi^*(h) &= \psi^*(V_{N=1}(\phi)h)V_N(\phi)\,, \end{aligned} \tag{3}$$

where the adjoint of these equations describes the behavior of ψ. The geometric meaning of (3) is evident: We think of ψ^* as creating a configuration in $\mathbf{M}$, and h as averaging over such configurations. The first equation in (3) states that both U and ψ^* act locally in $\mathbf{M}$. The second equation states that creating a single new configuration and then transforming the state vector by a diffeomorphism of $\mathbf{M}$ gives the same result as transforming $\mathbf{M}$ first and then creating the transformed new object, with the transformation law for individual configurations given by the action of $V_{N=1}(\phi)$.

Our general perspective is that for an indexed set of representations (2) to form a hierarchy, it is necessary and sufficient that ψ^* and ψ can be constructed obeying (3). We expect this general structure to occur not only for point particles, but also for extended objects such as vortex filaments or tubes. In this case the argument of ψ^* and ψ is a one-vortex Hilbert space vector, so that the creation and annihilation fields, even before averaging, do not depend on a single point in space but on a spatially extended configuration. Only $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ always have as their arguments individual points in space.

The bracket that the intertwining field obeys with elements of the Lie algebra (1) follows from (3):

$$\begin{aligned} [\rho(f), \psi^*(h)] &= \psi^*(\rho_{N=1}(f)h)\,, \\ [J(\mathbf{g}), \psi^*(h)] &= \psi^*(J_{N=1}(\mathbf{g})h)\,, \end{aligned} \tag{4}$$

where the bracket with ψ is given by the adjoint equations. Note that *only commutator brackets* occur here.

It is a straightforward, though lengthy, calculation to verify that canonical Bose and Fermi nonrelativistic fields satisfy (4). The interesting point is that one can begin with the N-particle Bose or Fermi representation of the current algebra, and *construct* intertwining fields that fulfill (4). It is a *consequence* of this construction that Bose fields obey canonical, equal-time commutation relations (+), and Fermi fields anticommutation relations (−),

$$\begin{aligned} [\psi(\mathbf{x}), \psi(\mathbf{y})]_\pm &= [\psi^*(\mathbf{x}).\psi^*(\mathbf{y})]_\pm = 0\,, \\ [\psi(\mathbf{x}), \psi^*(\mathbf{y})]_\pm &= \delta(\mathbf{x} - \mathbf{y})\,. \end{aligned} \tag{5}$$

As a further consequence of this construction, we obtain $\rho(\mathbf{x}) = m\psi^*(\mathbf{x})\psi(\mathbf{x})$ and $\mathbf{J}(\mathbf{x}) = (\hbar/2i)\{\psi^*(\mathbf{x}) \times [\nabla\psi(\mathbf{x})] - [\nabla\psi^*(\mathbf{x})]\psi(\mathbf{x})\}$, heretofore taken as the defining equations for the currents. Note again that the same Lie algebra holds for ρ and $\mathbf{J}$ in the Bose and Fermi cases.

We now construct explicit anyon fields obeying (3), anticipating that they will satisfy different brackets from Bose and Fermi fields. This is done as follows. First, we write the N-anyon representation of (3) for $\mathbf{M} = \mathbf{R}^2$, using the covering space of N-particle configuration space in the plane. Second, we make the representation concrete by introducing a way to realize an element of the covering space by N paths in the plane. Third, we use this to define ψ^* as a creation operator mapping $\mathcal{H}_N$ to $\mathcal{H}_{N+1}$. Finally, we state our results about ψ^* and ψ.

To write the N-anyon representation, we recall that a configuration is an *unordered* set γ of N distinct points in the plane: $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbf{R}^2$; the indexing of γ is arbitrary. The configuration space Δ_N is the set of all such γ. The topology of Δ_N that leads to anyon representations is described by its *fundamental group* which is the braid group B_N; an introduction to B_N in relation to anyon physics is given in [9]. A configuration γ together with a braid b labels an element of the *universal covering space* of Δ_N. We write $\tilde{\Delta}_N$ for this covering space, and $\tilde{\gamma} = (\gamma, b)$. In physical terms, the N points of γ label the positions of the anyons, while the braid b describes exactly how many times and in what order these anyons may have circled each other, starting from a reference configuration. As the latter is arbitrary, the labeling of elements of $\tilde{\Delta}_N$ by the pair (γ, b) is not unique but conventional; given γ, the element of $\tilde{\Delta}_N$ associated with the identity in B_N may be selected freely.

There are two ways of writing N-anyon wave functions so that the action of diffeomorphisms can be specified. If one lets Ψ depend only on the N points in γ, anyon statistics comes from an explicit phase in the operators $V(\phi)$. Alternatively, one can introduce $\tilde{\Psi}$ as a function of both γ and b. Then $\tilde{\Psi}$ satisfies a symmetry condition analogous to the familiar Bose or Fermi exchange symmetry. Also known as an equivariance condition, this just expresses the usual phase shift under exchange which defines the anyon statistics. It is expressed by requiring $\tilde{\Psi}$ to transform according to a representation $T(b)$ of B_N by complex numbers of modulus 1:

$$\tilde{\Psi}(\gamma, bb') = T(b')\tilde{\Psi}(\gamma, b)\,, \tag{6}$$

where bb' is the product of braids in B_N. Elsewhere we stress [10] that these ideas are *not* restricted to complex-valued wave functions and one-dimensional representations of B_N; quantum theories based on higher-dimensional, non-Abelian representations permit braid parastatistics (plektons). But we limit ourselves here to the usual anyon case where, when b is the braid for a single, counterclockwise exchange of two particles, $T(b) = \exp i\theta$. Denote by p the projection map $p(\tilde{\gamma}) = \gamma$. The obvious way that diffeomorphisms of the plane act on γ, $\phi\gamma = \{\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_N)\}$, lifts uniquely to $\tilde{\gamma} \in \tilde{\Delta}_N$ so that $p(\phi\tilde{\gamma}) = \phi p(\tilde{\gamma}) = \phi\gamma$. Then the N-anyon representation of (2) is given by [2,3]

$$U_N(f)\tilde{\Psi}(\tilde{\gamma}) = \exp[i\langle\tilde{\gamma}, f\rangle]\tilde{\Psi}(\tilde{\gamma}),$$
$$V_N(\phi)\tilde{\Psi}(\tilde{\gamma}) = \tilde{\Psi}(\phi\tilde{\gamma})\prod_{j=1}^{N}\sqrt{J_\phi(\mathbf{x}_j)}, \tag{7}$$

where $\langle\tilde{\gamma}, f\rangle = \sum_j f(\mathbf{x}_j)$ when $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, and where $J_\phi(\mathbf{x})$ is the Jacobian of ϕ at $\mathbf{x}$.

Next we introduce a concrete realization of $\tilde{\gamma}$ that assists in understanding the action of ϕ in $\tilde{\Delta}_N$. Write $\mathbf{x} \in \mathbf{R}^2$ in Cartesian coordinates as (x^1, x^2). Let γ be such that all the $\mathbf{x}_j$ have distinct values of their first coordinates: i.e., $x_j^1 \neq x_k^1$ for $j \neq k$. For such γ, consider a set $\Gamma = \{\Gamma_j\}$ of N continuous, non-self-intersecting and non-mutually-intersecting paths coming in from infinity, possibly circling some points, and terminating at the $\mathbf{x}_j$. We take all paths at infinity to extend in the negative x^2 direction, parallel to the x^2 axis. Then an element $\tilde{\gamma}$ can be identified with an equivalence class (homotopy class) $[\Gamma]$ of such paths, whose set of terminal points is γ. Given γ, we can make a *canonical choice* of $\tilde{\gamma}$ by letting all the paths be straight half lines parallel to the x^2 axis. Call this particular set of paths $\Gamma_0^{\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}}$, or Γ_0^γ; it is shown in Fig. 1. We associate this set of paths with the identity braid. Since the indexing of the $\mathbf{x}_j$ is arbitrary, we can label the paths and their terminal points so that $x_1^1 < x_2^1 < \cdots < x_N^1$ with Γ_j terminating at $\mathbf{x}_j$.

The key point is that diffeomorphisms of $\mathbf{R}^2$ act not only on configurations γ, but on sets of paths Γ, since these also lie in the plane. A diffeomorphism that is trivial at infinity respects (homotopy) equivalence among paths, so that it acts on $[\Gamma]$. Thus, for fixed γ, diffeomorphisms that leave γ unchanged (as a set) map the classes $[\Gamma]$ of paths terminating at γ into each other. For example, take a fixed pair of points $\{\mathbf{x}_1, \mathbf{x}_2\}$ in the plane. Consider the canonical paths $\Gamma_0^{\{\mathbf{x}_1,\mathbf{x}_2\}}$ constructed as in Fig. 1, terminating at $\{\mathbf{x}_1, \mathbf{x}_2\}$. Let ϕ be a diffeomorphism, trivial at infinity, that exchanges the points; i.e., $\mathbf{x}_2 = \phi(\mathbf{x}_1)$ and $\mathbf{x}_1 = \phi(\mathbf{x}_2)$. One way ϕ can act on the pair of paths $\Gamma_0^{\{\mathbf{x}_1,\mathbf{x}_2\}}$ is to map them to new paths as in Fig. 2 (imagine ϕ moving points only in the shaded region of the plane). Then we associate with this diffeomorphism the generator b_{12} in the braid group for a single counterclockwise exchange of the two points. To a diffeomorphism implementing one clockwise exchange of the points, we assign b_{12}^{-1}. A group homomorphism is defined in this manner, from the subgroup of diffeomorphisms that leave γ fixed as a set, onto the braid group. We denote the homomorphism by h_γ, and write $h_\gamma(\phi) = b$ for the braid associated with ϕ. The given realization provides a faithful mapping from B_N to classes of paths $[\Gamma]$. When b is the braid that takes $[\Gamma_0^\gamma]$ to $[\Gamma]$, we can write $T(\Gamma)$ in place of $T(b)$. Further explanation is given in [11].

Next we use this picture to define the anyon creation field ψ^*. To do this we need a consistent way to add an anyon at $\mathbf{x}$, not merely to an N-anyon configuration, but to an element of the N-anyon covering space. Doing this breaks equivalence among sets of paths, because many different points on the $(N+1)$-anyon covering space correspond to the introduction of a new anyon at $\mathbf{x}$. Our procedure is shown in Fig. 3. Given $[\Gamma_0^\gamma]$, and the additional point $\mathbf{x}$, define a new set of paths $\Gamma_{\mathbf{x}}^{\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}}$ by placing the point $\mathbf{x}$ in the plane among the

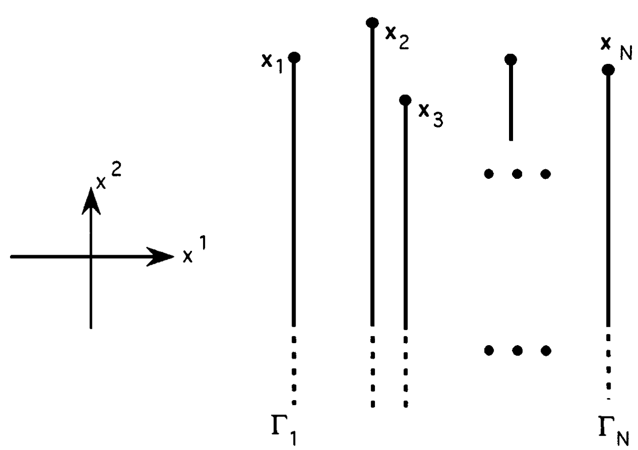

FIG. 1. For $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, a canonical choice Γ_0^γ of paths $\{\Gamma_j\}$ terminating at $\{\mathbf{x}_j\}$.

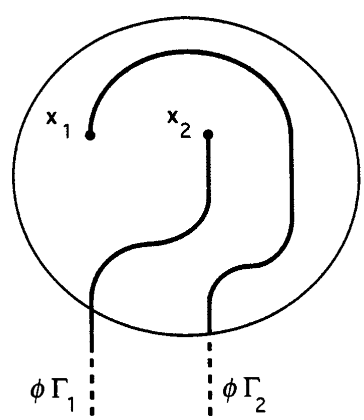

FIG. 2. A diffeomorphism implementing a counterclockwise exchange of two points labeled originally as in Fig. 1. Mirror image paths describe clockwise exchange.

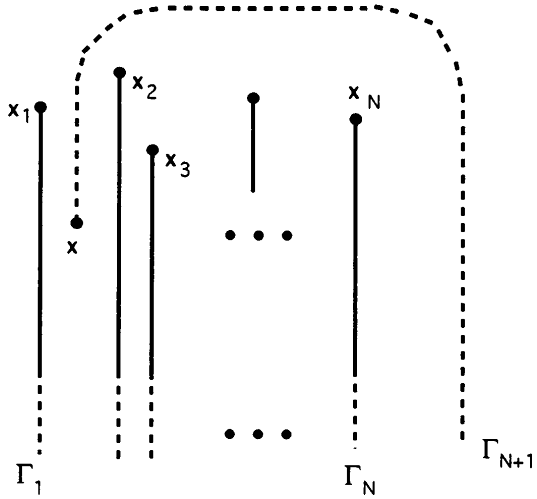

FIG. 3. An anyonic particle is created at $\mathbf{x}$, defining the element $\Gamma_{\mathbf{x}}^{\gamma}$ of the $(N + 1)$-anyon covering space.

N paths comprising Γ_0^{γ}, and drawing a new path Γ_{N+1} that terminates at $\mathbf{x}$, and comes in from infinity to the *right* of the N existing paths without intersecting them. This tells us a particular element of the $(N + 1)$-anyon covering space.

We now see from Figs. 1–3 why the q commutator will enter. Consider two points $\{\mathbf{x}_1, \mathbf{x}_2\}$. Creating an anyon first at $\mathbf{x}_2$, we obtain the path $\Gamma_1 = \Gamma_0^{\{\mathbf{x}_2\}}$, a vertical straight line terminating at $\mathbf{x}_2$. Creating the next anyon at $\mathbf{x}_1$ gives us paths in the class $[\Gamma_{\mathbf{x}_1}^{\{\mathbf{x}_2\}}]$ as in Fig. 2, corresponding to b_{12}. But if we first create an anyon at $\mathbf{x}_1$ and then at $\mathbf{x}_2$, we obtain the class $\Gamma_0^{\{\mathbf{x}_1, \mathbf{x}_2\}}$ associated with the identity braid. The relative phase $q = T(b_{12}) = \exp[i\theta]$ occurs in the two products of creation operators, where θ characterizes the anyons in the hierarchy.

To describe the action of the anyon annihilation and creation fields, let $\tilde{\Psi}$ be a sequence $(\tilde{\Psi}_N)$ of equivariant wave functions; if Γ is obtained from Γ_0^{γ} by the braid $b = h_{\gamma}(\phi)$, then $\tilde{\Psi}_N(\gamma, \Gamma) = T(b)\tilde{\Psi}_N(\gamma, \Gamma_0^{\gamma})$, or alternatively $\tilde{\Psi}_N(\gamma, \Gamma) = T(\Gamma)\tilde{\Psi}_N(\gamma, \Gamma_0^{\gamma})$. Then with $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, we set

$$\begin{aligned} [\psi(\mathbf{x})\tilde{\Psi}]_N(\gamma, \Gamma_0^{\gamma}) &= \tilde{\Psi}_{N+1}(\{\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{x}\}, \Gamma_{\mathbf{x}}^{\gamma}), \\ [\psi^*(\mathbf{x})\tilde{\Psi}]_N(\gamma, \Gamma_0^{\gamma}) &= \sum_{j=1}^{N} \delta(\mathbf{x} - \mathbf{x}_j)\tilde{\Psi}_{N-1}(\hat{\gamma}_j, \Gamma_0^{\hat{\gamma}_j}) \\ &\quad \times T^*(\Gamma_{\mathbf{x}}^{\hat{\gamma}_j}), \end{aligned} \quad (8)$$

where $\hat{\gamma}_j$ means that the point $\mathbf{x}_j$ is omitted from γ. Equations (8) extend from $\tilde{\Psi}_N(\gamma, \Gamma_0^{\gamma})$ to the (infinitely many) values of $\tilde{\Psi}_N(\gamma, \Gamma)$, using the equivariance property. From (7) and (8) one can check directly that (3) is satisfied. The infinitesimal generators ρ and J obey

$$\begin{aligned} [\rho(f), \psi^*(h)] &= \psi^*(mfh), \\ [J(\mathbf{g}), \psi^*(h)] &= \psi^*\left(\frac{\hbar}{2i}\{\mathbf{g} \cdot \nabla h + \nabla \cdot (\mathbf{g}h)\}\right), \end{aligned} \quad (9)$$

together with the adjoint equations for ψ. These *commutator* brackets hold independent of whether one has bosons, fermions, or anyons. Furthermore, we recover the operators $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ in terms of the anyon fields as the desired expressions as in the Bose and Fermi cases, with $\int(\tilde{\Psi}_N, \rho(\mathbf{x})\tilde{\Psi}_N)d^2x = Nm$.

Finally, we determine straightforwardly from (8) that the anyon fields obey q-commutation relations, where $q = T(b_{12})$. The q-deformed bracket is $[A, B]_q = AB - qBA$, where q is here a complex number of modulus 1. Then

$$\begin{aligned} [\psi(\mathbf{x}, t), \psi(\mathbf{y}, t)]_q &= [\psi^*(\mathbf{x}, t), \psi^*(\mathbf{y}, t)]_q = 0, \\ [\psi(\mathbf{y}, t), \psi^*(\mathbf{x}, t)]_q &= \delta(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (10)$$

Note that for the first two equations of (10) to be consistent when $q \neq \pm 1$, they should be interpreted as holding for ordered pairs $(\mathbf{x}, \mathbf{y})$ in a half space H of $M \times M$. In the complementary half space we have the $(1/q)$ bracket. The equation for $[\psi(\mathbf{x}, \psi(\mathbf{y})]_q$ is then consistent with the equation for $[\psi^*(\mathbf{x}), \psi^*(\mathbf{y})]_q$, since $|q| = 1$. The half space here is not a limitation, and its choice has no physical consequence; it is just an arbitrary boundary between sheets in the covering space. The third equation of (10) is written as indicated for $(\mathbf{x}, \mathbf{y}) \in H$; it may be written equivalently (using $\overline{q} = 1/q$) as $[\psi(\mathbf{x}, t), \psi^*(\mathbf{y}, t)]_{1/q} = \delta(\mathbf{x} - \mathbf{y})$. One can also verity that the brackets for ψ and ψ^* with ρ and $\mathbf{J}$ are in accordance with (4), using the algebraic identity

$$[AB, C]_- = A[B, C]_q + q[A, C]_{1/q}B \quad (11)$$

relating the ordinary commutator to the q commutator. The commutators of the field operators with the generators of the infinite-dimensional group hold for all values of $\mathbf{x}$ and $\mathbf{y}$, not merely in a half space.

As noted, we expect the methods of this Letter to generalize to still other hierarchies of diffeomorphism group representations; e.g., to extended objects like quantized vortex loops and tubes.

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