Numerical Tests of the Chiral Luttinger Liquid Theory for Fractional Hall Edges

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We report on microscopic numerical studies that support the chiral Luttinger liquid theory of the fractional Hall edge proposed by Wen. Our calculations are based in part on newly proposed and accurate many-body trial wave functions for the low-energy edge excitations of fractional incompressible states.

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The quantum Hall effect (QHE) can occur only when a disorder-free two-dimensional electron system has an incompressibility, i.e., a discontinuity in the chemical potential, at a magnetic field dependent density, $n^*(B)$. The incompressibility implies a gap for charged and neutral excitations in the bulk of the system, but the magnetic field dependence of the density at which the gap occurs requires the existence of gapless excitations localized at the edge of the system [1]. In equilibrium the current responsible for the orbital diamagnetism of the underlying two-dimensional electron system is carried at the edge and satisfies the thermodynamic identity,

$$\frac{\partial I}{\partial \mu} = c\,\frac{dn^*(B)}{dB}\,. \qquad (1)$$

In the *edge-state* picture [2], the quantum Hall effect follows from Eq. (1) when local equilibria are established on uncoupled edges. For the case of bulk incompressibilities at fractional Landau level filling factors (ν), which are of many-body origin, Wen's chiral Luttinger liquid (CLL) theory [3,4] of the low-energy edge excitations predicts non-Fermi-liquid effects, which have recently been of great interest [3–5]. The simplest version of this theory, and the one we test numerically here, applies to the edge of the incompressible ground states that occur at $\nu = 1/m$ for m odd [6].

The non-Fermi-liquid properties of one-dimensional fermion systems captured by the Luttinger model [7] arise from interactions between left-going and right-going particles in states close to the Fermi level. In a single-particle model, left-going and right-going states close to the Fermi energy in the QHE regime are localized on opposite edges of a sample with Hall bar geometry, and therefore interact weakly. However, Wen's theory predicts that in the fractional QHE regime, non-Fermi-liquid behavior arises *without interedge interactions.* We therefore consider a quantum Hall droplet (QHD) system for which electrons are confined to a finite area by a circularly symmetric external potential and for which a single circular edge exists. For such a system it follows [3,8,9] directly from the microscopic fractional QHE physics [6], which gives rise to the chemical potential jump at $\nu = 1/m$, that the low-energy neutral edge excitations for any m are in one-to-one correspondence with those of a chiral one-dimensional noninteracting fermion system that has single-particle states with only one sign of velocity. The noninteracting fermion system can in turn be mapped [7] to a 1D system of noninteracting chiral bosons; the bosons are the phonon modes of the edge. The same conclusion about the neutral excitations of a $\nu = 1/m$ edge can be reached by using a hydrodynamic approach [3,4] to derive a low-energy effective Hamiltonian expressed solely in terms of 1D charge densities obtained by integrating the 2D charge density along the coordinate perpendicular to the edge. The Hamiltonian is quantized by invoking a commutation relation between Fourier components of the 1D charge density,

$$[\rho(q), \rho(-q')] = \nu\,\frac{qL}{2\pi}\,\delta_{q,q'}\,. \qquad (2)$$

For $\nu = 1$ Eq. (2) can be derived microscopically, but for the fractional case it is assumed in order to satisfy Eq. (1) without altering the structure of the theory. This seemingly innocent introduction of the factor ν on the right-hand side of Eq. (2) is responsible for the non-Fermi-liquid behavior predictions [3–5] of CLL theory. The predictions follow from the expression for the electron field operator, which plays a central role in the theory, in terms of boson operators. This expression, quoted below, is the simplest choice that results in a field operator that satisfies the correct commutation relation with the density operator. For the fractional case, it has not been possible to fully justify this expression on the basis of microscopic theory. This situation motivates the extensive numerical tests reported in this article [10]. Our findings are in complete agreement with the CLL theory.

Our numerical calculations were performed for a QHD with a parabolic confinement potential. These model systems [11,12] are of direct relevance to transport [13–15] and capacitance [16] measurements in quantum dots in strong magnetic fields. Here we are interested principally in the edge excitations of QHD's, which are as large as possible in order to model the edge excitations of macroscopic incompressible states. In this model the single-particle state with angular momentum m has energy $(m+1)\Omega_0^2\ell^2$ where Ω_0 is the frequency characterizing the parabolic

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confining potential and $\ell = (\hbar c/eB)^{1/2}$ is the magnetic length. For a QHD the angular momentum plays the role that would be played by the linear momentum in units of $2\pi/L$ in a Hall bar geometry. For this microscopic model system the incompressible ground states $\Psi_0^m(N)$ associated with the $\nu = 1/m$ fractional QHE occur [8] for total angular momentum $M = M_0(m,N) = mN(N-1)/2$ with N being the number of particles in the QHD. The $m = 1$ incompressible state is a single Slater determinant in which single-particle states with $m = 0,1,\ldots,N-1$ are occupied.

In the language appropriate to the QHD the principal predictions of the CLL theory are the following: (i) For $M = M_0(m,N) + \Delta M$ the spectrum has a low-energy branch with many-particle eigenenergies given by $E = E_0 + \sum_l n_l e_l$ where $\sum_l l n_l = \Delta M$ and E_0 is the energy of the $M_0(m,N)$ incompressible state. This property is expected to be accurately satisfied for $\Delta M < N^{1/2}$, since the excitations are then well localized at the edge. Here n_l are the non-negative integers, which give the occupation numbers for the bosonic edge-wave angular momentum l and energy e_l. (ii) The electron creation operator is given by

$$\hat{\psi}^\dagger(\theta) = \sqrt{z}\,\exp[\hat{\phi}_+(\theta)]\exp[-\hat{\phi}_-(\theta)], \qquad (3)$$

where

$$\hat{\phi}_+(\theta) = \sum_l \sqrt{1/l\nu}\,[a_l^\dagger \exp(il\theta)], \qquad (4)$$

$a_l^\dagger$ is a boson creation operator, z is a constant that is not fixed in the CLL theory, $\phi_-(\theta) = [\phi_+(\theta)]^\dagger$, and for each particle number the boson operators act on the bosonic quantum numbers of the edge waves.

We first discuss our tests of the CLL theory predictions for the bosonic nature of the excitation spectrum of the edge of the QHD. Substantial arguments can be advanced in favor of this aspect of the CLL theory predictions from microscopic theory. In the case of the hard-core model of electron-electron interactions, the low-energy portion of the spectrum has no contributions from electron-electron interactions, the bosonization follows from analytic arguments, and $e_l = l\Omega_0^2\ell^2$ [8,9]. More generally, qualitative aspects of the bosonization of the low-energy portion of the spectrum follow from the adiabatic evolution of the spectrum with changing model interactions. For the QHD the $l = 1$ single-boson state corresponds microscopically to an excitation of the center of mass of all electrons from $M_{\rm c.m.} = 0$ to $M_{\rm c.m.} = 1$. Since the interaction energy of the electrons is independent of the center-of-mass state [17,18], it follows $e_1 = \Omega_0^2\ell^2$, independent of electron-electron interactions. For $l \neq 1$ e_l is interaction dependent; to test the bosonization for the physically realistic Coulomb interaction model we have determined the spectrum of finite-size QHD's by exact diagonalization of the many-particle Hamiltonian neglecting confinement. For parabolically confined QHD's the eigenstates are unchanged, and the subspace spectrum at total angular momentum M shifts rigidly by $M\Omega_0^2\ell^2$ when the confinement is introduced.

Figure 1 shows the spectra of the QHD for $\Omega_0 = 0$ as a function of M close to $M = M_0(3,N)$ and $M = M_0(1,N)$. Note that the interaction energy decreases as M increases because of the decrease in average two-dimensional electron density; in the QHD case interactions lower the boson energies because the charge spreads out in the direction perpendicular to the edge when the angular momentum is changed. Figure 1 shows that the bosonization law for the neutral edge-wave excitation spectrum, which is exact for the hard-core model, is still closely obeyed for the physically realistic Coulomb interactions. The bosonization is even more robust than would be expected *a priori* since it appears to hold even where $e_l \propto l$ fails.

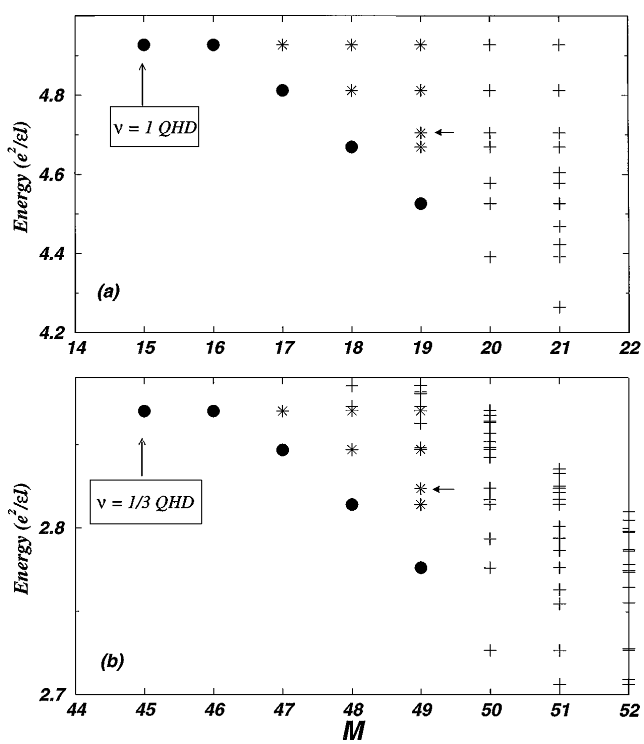

FIG. 1. Low-lying excitation spectra of a six-particle QHD neglecting confinement energies. (a) $\nu = 1$. The incompressible ground state occurs at $M = 15$. The solid dots show the interaction energies of the single-boson edge states, e_l ($e_1 = 0$). The highlighted asterisk at $M = 19$ shows the energy of the state with $n_2 = 2$. Its energy differs from $2e_2$ by 3.3%. The sequence of states that appear along horizontal lines in this figure corresponds microscopically to increasing values of the center-of-mass angular momentum and, in the boson picture, to increasing values of n_1. (b) $\nu = 1/3$. The incompressible ground state occurs at $M = 45$. The similarity with the $\nu = 1$ spectrum in (a) is clear despite the appearance for $\Delta M \geq 3$ of other states representing bulk excitations. These states are expected to move to relatively higher energies for large N. If higher LL's were included in the calculation, the same incursion of bulk excitations for values of ΔM near N could occur in the $\nu = 1$ case at small enough values of $\hbar\omega_c$. In the $\nu = 1/3$ case the energy of the highlighted $n_2 = 2$ state differs from $2e_2$ by 0.3%.

More extensive analyses of the spectrum are possible and have been completed previously *only* for the edge excitations of a $\nu = 1$ QHD where exact diagonalization techniques can be applied for much larger numbers of electrons [19]. A set of microscopic operators have been proposed by Stone [19] to generate the edge-wave spectrum in the $\nu = 1$ case: $S^\dagger_{\Delta M} = \sum_m [(m + \Delta M)!/m!]^{1/2} c^\dagger_{m+\Delta M} c_m$ where $c^\dagger_m$ and c_m are the electron creation and annihilation operators. Recently, Oaknin *et al.* [20] identified a modified set of operators, $J^\dagger_{\Delta M} = \sum_m [m!/(m + \Delta M)!]^{1/2} c^\dagger_{m+\Delta M} c_m$, which, when acting upon the $\nu = 1$ QHD, do not alter the center-of-mass state of the electrons. It was demonstrated by Oaknin *et al.* that their operators generate the single-boson excitations of the $\nu = 1$ QHD more accurately than those of Stone [20]. More generally we find by comparing with exact eigenstates that for a given ΔM, the excitations with large bosonic occupation numbers are more accurately generated by the $S^\dagger$ operators than by the $J^\dagger$ operators, while the comparison is reversed for small bosonic occupation numbers. In the limit $\Delta M < N^{1/2}$ the states generated by the two sets of operators become equivalent.

In order to access large N for a fractional QHD, following Ref. [8] we propose a set of microscopic many-body wave functions for low-lying excited states of fractional QHD's,

$$D^{m-1} \prod_l (J^\dagger_l)^{n_l} |\Psi^1_0(N)\rangle \,, \tag{5}$$

where $\sum_l l n_l = \Delta M$ and D is the Vandermonde determinant [8] Jastrow factor that relates different Laughlin QHD states $[\Psi^m_0(N) = D^2 \Psi^{m-2}_0(N)]$. (As discussed above, for some states greater accuracy at finite N can be achieved on substituting $J^\dagger$ by $S^\dagger$.) These trial wave functions have the following properties: (i) they have the appropriate value of the angular momentum; (ii) for $\Delta M = 0$ they reduce to Laughlin's approximation for the incompressible state many-body wave function; (iii) they are exact eigenstates with zero interaction energy in the case of the hard-core model Hamiltonian; and (iv) the single-boson states are eigenstates of center-of-mass angular momentum with eigenvalue 0 for $\Delta M \neq 1$ and eigenvalue 1 for $\Delta M = 1$. If the set $S^\dagger$ is used instead of $J^\dagger$, the final property still holds but only in the limit $\Delta M < N^{1/2}$. Table I shows the absolute value of the overlaps between the trial single-boson state wave functions and the exact ones for the $\nu = 1/3$ QHD for up to six particles. We see that the excited state wave functions ($\Delta M \geq 2$) tend to be, if anything, more accurate than the Laughlin trial wave function for the ground state ($\Delta M = 0$). (The bracketed overlap values were calculated using the Stone operators.) It is important to realize that the direct application of the Oaknin *et al.* operators to the $\nu = 1/3$ Laughlin state would not generate identical states since $[D^{m-1}, J^\dagger_{\Delta M}] \neq 0$; the resulting states would lack properties (iii) and (iv) from the list above, and they are very poor approximations to the true states. On the other hand, $[D^{m-1}, S^\dagger_{\Delta M}] = 0$.

TABLE I. Absolute value of the overlaps between the exact single-boson state (see text) and the trial states $D^2 J^\dagger_{\Delta M} |\Psi^1_0(N)\rangle$ for up to six particles (in brackets using $S^\dagger_{\Delta M}$). The high overlaps confirm the validity of the mapping (through D^2) between the single-boson state excitations of a $\nu = 1$ QHD and those of a $\nu = 1/3$ QHD.

ΔM	$N = 4$	$N = 5$	$N = 6$
0	0.9788 (0.9788)	0.9850 (0.9850)	0.9819 (0.9819)
1	0.9788 (0.9788)	0.9850 (0.9850)	0.9819 (0.9819)
2	0.9768 (0.9660)	0.9715 (0.9647)	0.9790 (0.9743)
3	0.9906 (0.9167)	0.9736 (0.9296)	0.9725 (0.9429)
4	0.9940 (0.7489)	0.9970 (0.8373)	0.9701 (0.8637)
5		0.9862 (0.6360)	0.9819 (0.7274)
6			0.9820 (0.5306)

We turn finally to our numerical tests of the less fully justified CLL theory expression for the electron field operator. We report here only results for the squared matrix elements connecting the ground state of the N electron system with the ground state and low-lying bosonic states of the $N + 1$ particle system at $\nu = 1/3$, $|\langle \Psi^3_{\{n_l\}}(N + 1)| \psi^\dagger(\theta) |\Psi^3_0(N)\rangle|^2$. The values predicted for these squared matrix elements by the CLL theory can easily be computed from Eq. (3). The predicted matrix elements for $\Delta M = 0$, 1, 2, 3, 4 are listed in Table II and compared with those calculated microscopically [21] for $N = 6$, 7, and 8. For each case the matrix elements have been normalized to the ground-state–to–ground-state matrix element z [22]. Since the angular momentum difference between the N and $N + 1$ particle states is $M_0(3, N + 1) + \Delta M - M_0(3, N) = 3N + \Delta M$, only the part of the electron creation operator proportional to $c^\dagger_{3N+\Delta M}$ contributes to the microscopic matrix element. The agreement between the CLL results for these matrix elements and the microscopic calculations is excellent. It appears from our extrapolation that the CLL theory becomes exact for $N \to \infty$. The non-Fermi-liquid power law properties predicted by the CLL theory depend both on the predictions for these matrix elements and on having dispersionless propagation of edge modes, i.e., $e_l = cl$. In that case the total spectral weight at an energy cl above the chemical potential is proportional to the sum of the squared matrix elements for $\Delta M = l$. It follows from Eq. (3) that this sum equals

$$A_{\Delta M} = z \, \frac{(\Delta M + m - 1)!}{(\Delta M)!\,(m - 1)!} \,. \tag{6}$$

For $\Delta M \gg m - 1$ the right-hand side of Eq. (6) approaches $(\Delta M)^{m-1}/(m - 1)!$. This gives the power law dependence of the spectral weight at low energies, which enters into various physical properties.

In conclusion, our microscopic numerical studies strongly support the chiral Luttinger liquid theory of the fractional Hall edge proposed by Wen and add to the motivation for experiments that can probe the low-energy excitations of incompressible fractional Hall states.

TABLE II. Microscopic and CLL theory spectral weights for the $\nu = 1/3$ QHD state in units of the ground-state–to–ground-state matrix element. Results are shown for $N = 6$, 7, 8 and are extrapolated to $N \to \infty$. The bracketed CLL theory results are for $\nu = 1$ where the squared matrix elements sum to 1 for each ΔM. For $\nu = 1/3$ each matrix element is increased by a factor of 3^k where $k = \sum_l n_l$. The increase in the spectral weights moving away the Fermi level is due to the increasing boson occupation numbers.

		$\lvert\langle\Psi^3_{\{n_l\}}(N+1)\rvert c^\dagger_{3N+\Delta M}\lvert\Psi^3_0(N)\rangle\rvert^2$				
ΔM	$\{n_l\}$	$N = 6$	$N = 7$	$N = 8$	$N \to \infty$	CLL theory
0	{0000}	1.000	1.000	1.000	1.000	1 (1)
1	{1000}	2.714	2.750	2.778	2.998	3 (1)
2	{2000}	3.877	3.953	4.012	4.473	9/2 (1/2)
	{0100}	1.322	1.343	1.358	1.445	3/2 (1/2)
3	{3000}	3.877	3.953	4.012	4.473	9/2 (1/6)
	{1100}	3.913	3.986	4.041	4.453	9/2 (1/2)
	{0010}	0.939	0.943	0.946	0.983	1 (1/3)
4	{4000}	3.047	3.088	3.121	3.360	27/8 (1/24)
	{2100}	6.024	6.131	6.209	6.710	27/4 (1/4)
	{1010}	2.828	2.852	2.869	2.966	3 (1/3)
	{0200}	1.048	1.058	1.064	1.101	9/8 (1/8)
	{0001}	0.830	0.811	0.797	0.725	3/4 (1/4)

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