Caracciolo *et al.* Reply: As Patrascioiu and Seiler [1] note, there are two *very different* limits that can be taken in a two-dimensional σ model: (a) $\beta \to \infty$ at *fixed* $L < \infty$, or (b) $\beta \to \infty$ and $L \to \infty$ such that the ratio $x \equiv \xi(\beta, L)/L$ is held fixed. Limit (b) is the one relevant to finite-size scaling, while perturbation theory is clearly valid in limit (a). The deep question is whether the perturbation theory derived from the study of limit (a) is *also* correct in the double limit obtained by first taking limit (b) and then taking $x \to \infty$. The conventional wisdom says *yes*: indeed, this or a similar interchange of limits underlies the conventional derivations of asymptotic freedom. Patrascioiu and Seiler say *no*: they suspect that asymptotic freedom is false [2]. At present, no rigorous proof is available to settle this question one way or the other.

Our analysis [3] of our Monte Carlo data is based on finite-size scaling [4–6], i.e., limit (b). Thus, at each *fixed* $x \equiv \xi(\beta, L)/L$, we ask whether the ratios $\mathcal{O}(\beta, 2L)/\mathcal{O}(\beta, L)$ have a good limit as $L \to \infty$, and we attempt to evaluate this limit numerically in the usual way: namely, we evaluate the ratios over a wide range of L (from 32 to 256), and we ask whether these ratios appear to be converging to a limit as L grows. We find, in fact, that the ratios are *constant* within error bars for $L \gtrsim 64-128$ (depending on the value of x). Of course, it is *conceivable* that this apparent limiting value is a deception—i.e., a "false plateau"—and that at much larger values of L the ratio will change dramatically. We acknowledge as much in the penultimate paragraph of our Letter. This caveat is not special to our work, but is inherent in *any* numerical work which attempts to evaluate a limit (here $L \to \infty$) by taking the relevant parameter *almost* to the limit (here L large but finite).

In any case, there is no evidence that this perverse scenario in fact occurs. The corrections to scaling in our data are very weak—less than 2% even at $L = 32$, and a fraction of a percent or smaller for $L \gtrsim 64-128$—and are perfectly consistent with a behavior of the form

$$\mathcal{O}(\beta, 2L)/\mathcal{O}(\beta, L) = F_{\mathcal{O}}(x) + G_{\mathcal{O}}(x)/L^2 + \cdots, \quad (1)$$

where the correction term $G_{\mathcal{O}}$ is negative for $0.3 \lesssim x \lesssim 0.7$ and is perhaps slightly positive for $x \gtrsim 0.7$. If all hell breaks loose for larger L—as the Patrascioiu-Seiler scenario would require—we certainly see no hint of it at $L \leq 256$.

Patrascioiu-Seiler also note that our Monte Carlo data at $x \gtrsim 0.7$ agree well with the two-loop perturbative prediction, shown as a dotted curve in Fig. 2 of [3]. But this does not mean that we are *assuming* asymptotic scaling (whether explicitly or implicitly). Quite the contrary: our data at $x \gtrsim 0.7$ constitute a (weak) *test* of asymptotic scaling. The same point (β, L) may well lie within the range of validity (to some given accuracy) of two distinct expansions. The fact that our data points at large x are consistent with finite-volume perturbation theory [limit (a)] does not constitute evidence against their *also* being consistent with nonperturbative finite-size scaling [limit (b)].

Of course, since our Monte Carlo data for $F_{\mathcal{O}}(x)$ at $x \gtrsim 0.7$ do in fact agree closely with the two-loop perturbative formula (to within about 1%), and our data for $\mathcal{O}(\beta, L)$ also agree well with the fixed-L perturbation expansion (to within a few percent), it is then inevitable that our extrapolated values $\xi_\infty(\beta)$ at the largest values of β will be consistent with asymptotic scaling, in the sense that $\xi_\infty(\beta)/[e^{2\pi\beta/(N-2)}\beta^{-1/(N-2)}]$ will be roughly constant. However, it is by no means inevitable that this constant value will agree with the Hasenfratz-Maggiore-Niedermayer prediction to within 4%. It seems to us that this apparent coincidence is significant evidence in favor of the asymptotic-freedom picture.

Finally, Patrascioiu and Seiler [7] have found an unusual boundary condition for which the $L \to \infty$ limit of the perturbative coefficients *disagrees* with those obtained from the same limit in periodic boundary conditions. Since the two boundary conditions should agree in the limit $L \to \infty$ at any *fixed* $\beta < \infty$, it follows that *for at least one of the two boundary conditions* the $L \to \infty$ limit fails to commute with perturbation expansion in powers of $1/\beta$. This is troubling, but it does not tell us *which* of the two boundary conditions is at fault. It is quite possible that the two limits *do* commute in periodic boundary conditions—as the conventional wisdom asserts—but not in Patrascioiu-Seiler's unusual boundary condition. Nevertheless, this example shows that the justification of the conventional wisdom—if indeed it is true—will be considerably more subtle than was heretofore believed.

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