Quantum Algebraic Nature of the Phonon Spectrum in ^{4}He

M. R-Monteiro* and L. M. C. S. Rodrigues†
Centro Brasileiro de Pesquisas Físicas (CBPF), Rua Dr. Xavier Sigaud, 150, 22290-180 Rio de Janeiro, Brazil

S. Wulck‡
Instituto de Física, Universidade Federal do Rio de Janeiro, Cidade Universitária-Ilha do Fundão, 21945-970 Rio de Janeiro, Brazil
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We propose that the phonons in ^{4}He obey a q deformation of the Heisenberg algebra and we give an algebraic interpretation for the polynomial expansion of the small momenta phonon dispersion relation. Comparison with C_V experimental data shows that our spectrum reproduces the experimental one with a less than 5% discrepancy.

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The superfluid properties of ^{4}He [1] are well described by Landau theory [2]; nevertheless, even for temperatures as low as 1 K there are still unsolved discrepancies between theory and experiment. In Landau theory, the superfluidity follows from phonon and roton elementary excitations [3]. The anomalous dispersion of the phonon spectrum in ^{4}He, $\omega(p) = c_0 p(1 - \gamma p^2)$ (c_0 is the sound velocity), was theoretically derived [4] and γ estimated to be positive. On the other hand, data from ^{4}He specific heat measurements were fit by a different expression for the dispersion of the phonon spectrum and give a negative γ for most values of the pressure [5,6]. Negative γ leads to an unstable phonon spectrum, which is confirmed by experimental measurements of phonon lifetime in scattering of neutrons [7]. In this Letter we show that this difficulty can be overcome if we treat the phonons as bosonic q oscillators [8]. Using A, B, and D values experimentally determined by fitting the low-temperature phonon specific heat

$$C_V^{\text{phonon}} = AT^3 + BT^5 + DT^7, \tag{1}$$

with measured specific heat data of ^{4}He [6] at the temperature range $0.14 \le T \le 0.86$, our model leads to unstable phonons for all the analyzed values of the pressure.

Bosonic q oscillators [8] are a generalization of the Heisenberg algebra obtained by introducing a deformation parameter q. For $q > 1$ [9], an ideal q gas presents Bose-Einstein condensation and the specific heat exhibits a λ-point discontinuity [10], two features connected to superfluidity [11]. On the other hand, there have been interesting indications that the continuum descriptions of physical quantities break down both in a convergent fluid flow [12] and, more recently, in superfluid ^{4}He [13]. A similar breakdown has been observed in connection with deformed algebras [14], and we are led to think that they might have a role to play in the study of superfluidity.

Let us then consider the algebra generated by a, $a^\dagger$ and N satisfying

$$\begin{aligned} [N, a^\dagger] &= a^\dagger, \qquad [N, a] = -a\,, \\ aa^\dagger - q^{-1}a^\dagger a &= q^N \qquad (q \in \mathbb{R})\,. \end{aligned} \tag{2}$$

Assuming that a and $a^\dagger$ are mutually adjoint, $N = N^\dagger$, and the spectrum is nondegenerate, the following representations of (2) were obtained [15] for $q > 1$:

$$\begin{aligned} a^\dagger|n\rangle &= q^{\nu_0/2}[n + 1]^{1/2}|n + 1\rangle\,, \\ a|n\rangle &= q^{\nu_0/2}[n]^{1/2}|n - 1\rangle\,, \\ N|n\rangle &= (\nu_0 + n)|n\rangle\,, \end{aligned} \tag{3}$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$ and ν_0 is a real free parameter which goes to zero when (2) becomes the usual Heisenberg algebra ($q \to 1$). Note that only when $\nu_0 = 0$, N is the usual particle number operator for the normalized vector state $|n\rangle$; otherwise, the particle number operator is $\hat{N} = N - \nu_0$ and ν_0 is a parameter that classifies the inequivalent representations of the algebra (2) [15–17].

Generalizing previous results obtained for $\nu_0 = 0$ [18], in the Fock space spanned by the vectors $|n\rangle$, we can express the above deformed oscillators in terms of the standard bosonic ones, b and $b^\dagger$, according to

$$\begin{aligned} a &= q^{\nu_0/2}\left(\frac{[N + 1 - \nu_0]}{N + 1 - \nu_0}\right)^{1/2} b\,, \\ a^\dagger &= q^{\nu_0/2} b^\dagger \left(\frac{[N + 1 - \nu_0]}{N + 1 - \nu_0}\right)^{1/2}, \end{aligned} \tag{4}$$

and it can be easily shown that

$$aa^\dagger = q^{\nu_0}[N + 1 - \nu_0], \quad a^\dagger a = q^{\nu_0}[N - \nu_0]\,, \tag{5}$$

where $N - \nu_0 = b^\dagger b$. This shows that bosonic q oscillators, in arbitrary representations ν_0 and for real $q > 1$, can be reinterpreted as standard bosonic oscillators.

We propose that the phonons in ^{4}He are described by a q gas. Considering that our model will be compared

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TABLE I. Values of c_0, α, δ, and q resulting from the least-squares fits of the specific heat data with the expression $C_V = \tilde{A}T^3 + \tilde{B}T^5 + \tilde{D}T^7 + C_V^{\rm roton}$, for samples 6–9 in analysis 2 of Ref. [6]; the roton data are those of [6].

Sample	$\tilde{A}/10^4$ (erg/mol K^4)	$\tilde{B}/10^4$ (erg/mol K^6)	$\tilde{D}/10^4$ (erg/mol K^8)	V (cm^3)	$c_0/10^4$ (cm/sec)	$\alpha/10^{38}$ (g^{-2} cm^{-2} sec^2)	$\delta/10^{19}$ (g^{-1} cm^{-1} sec)	q
6	84.42	−49.8	83	27.5790	2.2854	2.1	1.7745	3.5090
7	69.3	−36.10	67	26.9650	2.4177	2.7178	1.9361	4.3956
8	57.77	−25.4	49	26.4240	2.5501	3.0873	2.0134	4.8778
9	49.85	−18.8	38	25.9760	2.6625	3.3821	2.0700	5.2490

with the experimental results at the temperature range $0.14 \leq T \leq 0.86$ [6], where the rotons contribution is at most 0.5% of the total specific heat [6], they will be treated as usual [3]. We take for the phonon gas the Hamiltonian

$$H = \sum_i \omega_i a_i^\dagger a_i = \sum_i \omega_i([N_i] - q^{N_i} C), \quad (6)$$

where $C = q^{-N}([N] - a^\dagger a)$ is a Casimir operator of the algebra (2) and in the representations (3) one has

$$C|n\rangle = q^{\nu_0}[\nu_0]\,|n\rangle. \quad (7)$$

In (6) a_i and $a_i^\dagger$ are the annihilation and creation operators, respectively, of particles in levels i with energy ω_i and N_i is the number operator of particles in levels i plus ν_0^i, which we are assuming level dependent.

As the partition function factorizes for the above system the canonical potential is

$$\Omega = -\frac{1}{\beta}\sum_i \ln \sum_{n=0}^{\infty} e^{-\beta\omega_i q^{\nu_0^i}[n]}, \quad (8)$$

where $\beta = (k_B T)^{-1}$, with k_B the Boltzmann constant.

The phonon anomalous dispersion relation is $\omega(p) = c_0 p(1 - \alpha p^2)$, with c_0 the velocity of sound, and we propose the dispersion relation

$$\nu_0(p) = \frac{\delta^2}{\theta}p^2 = \frac{p^2/2m}{E_\lambda}, \quad (9)$$

with δ an algebraic dimensional constant, $[\delta] =$ g^{-1} cm^{-1} sec, and $q = e^\theta$. As a consequence of the dimensionlessness of $\nu_0(p)$ it appears in (9) an energy scale, E_λ, that we take as $E_\lambda = k_B T_\lambda$, where T_λ is the temperature at which liquid ^{4}He undergoes a transition and becomes superfluid. Moreover, it seems natural to take $m = m_{^4\rm He}$ since we have for $\nu_0(p)$ the nonrelativistic classical dispersion law. For small phonon momenta we can expand our energy-momentum relation as

$$\begin{aligned} q^{\nu_0(p)}\omega(p) &= e^{\delta^2 p^2} c_0 p(1 - \alpha p^2) \\ &= c_0 p[1 - (\alpha - \delta^2)p^2 \\ &\quad - (\alpha\delta^2 - \tfrac{1}{2}\delta^4)p^4 - \cdots]. \end{aligned} \quad (10)$$

We are thus presenting an algebraic interpretation to the usually *ad hoc* introduced small momenta phonon dispersion relation [3,5,6].

It follows from a straightforward calculation that the low-temperature q-phonon specific heat per mole is given by

$$C_{V,q}^{\rm phonon} = \tilde{A}T^3 + \tilde{B}T^5 + \tilde{D}T^7 + \tilde{G}T^9 + \cdots, \quad (11)$$

where

$$\tilde{A} = \frac{2k_B^4 V}{\pi^2\hbar^3 c_0^3}\,\omega^{(3)}, \quad \tilde{B} = \frac{15k_B^6(\alpha - \delta^2)V}{\pi^2\hbar^3 c_0^5}\,\omega^{(5)}, \quad \tilde{D} = \frac{28k_B^8(\frac{7}{2}\alpha^4 + 4\delta^2 - 7\alpha\delta^2)V}{\pi^2\hbar^3 c_0^7}\,\omega^{(7)},$$

$$\tilde{G} = \frac{15k_B^{10}(-81\delta^6 + 110\alpha^3 + 243\alpha\delta^4 - 270\delta^2\alpha^2)V}{2\pi^2\hbar^3 c_0^9}\,\omega^{(9)}, \quad (12)$$

with V the molar volume and

$$\omega^{(m)} = \int_0^\infty dy\, y^{m-1}\frac{\sum_{n=0}^\infty [n]e^{-y[n]}}{\sum_{n=0}^\infty e^{-y[n]}}. \quad (13)$$

Using its usual dispersion relation $\omega_r(p) = \Delta + (p - p_0)^2/2\mu$, where Δ is the energy gap, p_0 is the position of the energy minimum and μ is the effective mass of the roton; the roton contribution to the molar specific heat is

$$\begin{aligned} C_V^{\rm roton} &= \frac{2V\mu^{1/2}p_0^2\Delta^2}{(2\pi)^{3/2}\hbar^3 k_B^{1/2}T^{3/2}} \\ &\quad \times [1 + k_BT/\Delta + \tfrac{3}{4}(k_BT/\Delta)^2]e^{-\Delta/k_BT}, \end{aligned} \quad (14)$$

and the total specific heat is

$$C_V = C_{V,q}^{\rm phonon} + C_V^{\rm roton}. \quad (15)$$

Taking for the coefficients $\tilde{A}$, $\tilde{B}$, and $\tilde{D}$ the least-squares fits for A, B, and D in (1) [6] of the measured specific heat data (analysis 2 in Ref. [6]) and $\tilde{G} = 0$, we obtain for q, α, δ, and c_0 the results listed in Table I. The values of q are derived from

$$\ln q = 2m_{^4\rm He}\delta^2 k_B T_\lambda, \quad (16)$$

which is a consequence of (9), and c_0, α, and δ from relations (12). As the very large errors in the T^7 coefficients for the samples 10–16 [6] lead to a high

TABLE II. In the upper row, we repeat the values of Table I for sample 6. In the lower one, we have the values for c_0, α, δ, and q obtained taking for $\tilde{A}$, $\tilde{B}$, $\tilde{D}$, and $\tilde{G}$ values that reproduce, within 5% accuracy, the curve resulting from least-squares fit of C_V data [6] with the expression $C_V = AT^3 + BT^5 + DT^7 + C_V^{\mathrm{roton}}$.

Sample	$\tilde{A}/10^4$ (erg/mol K^4)	$\tilde{B}/10^4$ (erg/mol K^6)	$\tilde{D}/10^4$ (erg/mol K^8)	$\tilde{G}/10^4$ (erg/mol K^{10})	V (cm^3)	$c_0/10^4$ (cm/sec)	$\alpha/10^{38}$ (g^{-2} cm^{-2} sec^2)	$\delta/10^{19}$ (g^{-1} cm^{-1} sec)	q
6	84.42	-49.8	83	0	27.5790	2.2854	2.1	1.7745	3.5090
6	80	-21	83	-67.5	27.5790	2.3209	3.4484	1.9815	4.7839

inaccuracy in the derivation of expression (16), we restrict our analysis to the samples 6–9 [6].

In Table I we see that the values of q increase with the pressure, and that the values of c_0 are around 4% lower than the directly measured sound velocities [19]. These results are obtained by least-squares fits of the specific heat data [6] with the expression (15), considering terms up to T^7 in $C_{V,q}^{\mathrm{phonon}}$. Since in our model higher powers of T are relevant, in the second row of Table II we show the results obtained considering terms up to T^9 in (11) and taking $\tilde{A} = 80 \times 10^4$ erg/mol K^4, $\tilde{B} = -21 \times 10^4$ erg/mol K^6, $\tilde{D} = 83 \times 10^4$ erg/mol K^8, and $\tilde{G} = -67.5 \times 10^4$ erg/mol K^{10}. These values reproduce, within 5% accuracy, the curve resulting from least-squares fit of C_V data for sample 6 [6], with $C_V = AT^3 + BT^5 + DT^7 + C_V^{\mathrm{roton}}$ (see Fig. 1). We see that the c_0 value is then more in accordance with the experimental one. We note that for a given value of q, c_0 and the parameters α and δ are obtained directly from the values of $\tilde{A}$, $\tilde{B}$, and $\tilde{D}$ through relations (12). The coefficient $\tilde{G}$ of T^9 is crucial to show the consistency of our model. In fact, with the values of α, δ, and c_0 in the lower row of Table II, the coefficient $\tilde{G}$ calculated from the last relation (12) is equal to -67.5×10^4 erg/mol K^{10}.

In summary, considering the phonons in ^{4}He as being described by a quantum q gas in a special representation of the Heisenberg algebra, we have shown the q-algebraic nature of the polynomial expansion of the small momenta phonon dispersion relation. Moreover, our estimated values of c_0 are in good agreement with the directly measured sound velocities. To test the present model we have compared it with the available experimental data: Our spectrum reproduces the experimental one for the entire $0.14 \leq T \leq 0.86$ range, within less than 5% discrepancy. Finally, we would like to stress that as a consequence of the proposed dispersion relation (10), with only two free parameters (q, ν_0) we have been able to fit the experimental data with the three coefficients $\tilde{B}$, $\tilde{D}$, and $\tilde{G}$ in the specific heat expansion (11).

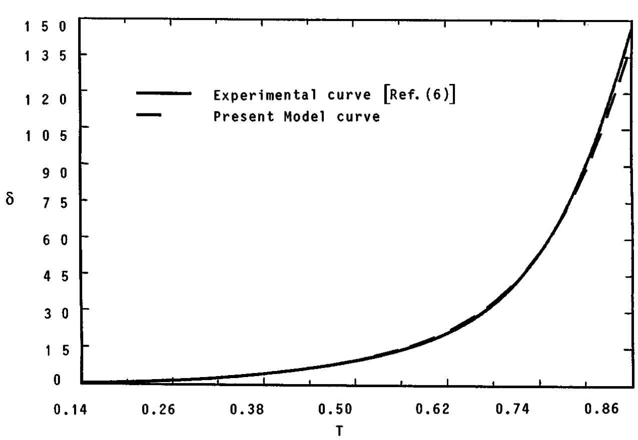

FIG. 1. Specific heat of ^{4}He. Comparison of the curve obtained in our model with the one resulting from least-squares fit of C_V data for sample 6 of analysis 2 [6].

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*Electronic address: RMONT@CBPFSU1.CAT.CBPF.BR
†Electronic address: LIGIA@CBPFSU1.CAT.CBPF.BR
‡Electronic address: STENIOW@IF.UFRJ.BR

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