High Density QCD with Static Quarks

T. Blum,* J. E. Hetrick, and D. Toussaint
Department of Physics, University of Arizona, Tucson, Arizona 85721
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We study lattice QCD in the limit that the quark mass and chemical potential are simultaneously made large, resulting in a controllable density of quarks which do not move. This is similar in spirit to the quenched approximation for zero density QCD. In this approximation we find that the deconfinement transition seen at zero density becomes a smooth crossover at very small density (possibly for any nonzero density), and that at low enough temperature chiral symmetry remains broken at all densities.

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Lattice QCD with a nonzero density of quarks is a difficult problem due to the fact that the fermion determinant is not a positive real number, and thus cannot be used as a weight for generating configurations by Monte Carlo methods [1]. A further technical difficulty is that with Kogut-Susskind quarks the fermion matrix is badly ill conditioned at nonzero chemical potential [2], making simulations even more difficult. At present, only crude results on very small lattices are available [3–5]. Faced with these difficulties, we may consider approximations which hopefully capture some of the essential features of the physics. Here we present a study of QCD at arbitrary quark density in an approximation where the dynamics of the quarks has been removed. This is analogous to the quenched approximation at zero density, an approximation which has provided considerable insight into the nature of QCD.

Our idea is to take simultaneously the limits of infinite quark mass and infinite chemical potential while the density of quarks remains fixed. This leaves us with quarks that can be present or absent at each lattice site, but which do not move in the spatial directions owing to their infinite mass. The result is a much simpler fermion determinant such that gauge variables can be updated to equilibrium in the background of a prescribed density of quarks, with little more difficulty than updating in the quenched approximation. (The simultaneous limit of $\mu \to \infty$ and $m \to \infty$ has also been suggested for QED with Wilson quarks in Ref. [6].)

The general idea of studying the problem in simple approximations is not new. DeGrand and DeTar have studied an extension of the three-dimensional Potts model with an imaginary magnetic field, which has similar symmetry breaking as in QCD, and might be expected to lie in the same universality class [7]. Satz has used a lowest order hopping parameter expansion on $8^3 \times 3$ lattices [8], which is also an approach based on very heavy quarks.

With a chemical potential included, the lattice Dirac operator using Kogut-Susskind quarks is

$$M(x,y) = 2am_q\delta_{x,y} + \sum_{\nu=1,2,3}[U_\nu(x)\eta_\nu(x)\delta_{x+\hat{\nu},y} - U_\nu^\dagger(y)\eta_\nu(y)\delta_{x-\hat{\nu},y}] + [e^{\mu a}U_t(x)\eta_t(x)\delta_{x+\hat{t},y} - e^{-\mu a}U_t^\dagger(y)\eta_t(y)\delta_{x-\hat{t},y}]. \tag{1}$$

Taking limits $m \to \infty$ and $\mu \to \infty$ simultaneously leaves $2ma$ along the diagonal and the forward hopping terms $e^{\mu a}U_t$. Each spatial point is decoupled from all others, and the fermion determinant is just a product of easily computed determinants on each static would line:

$$\det(M) = \prod_{\vec{x}} e^{\mu a n_c n_t}\det(P_{\vec{x}} + C\mathbf{1}). \tag{2}$$

Here $P_{\vec{x}}$ is the Polyakov loop at spatial site $\vec{x}$, n_c is the number of colors, and n_t is the number of time slices. The coefficient of the unit matrix C is $(2ma/e^{\mu a})^{n_t}$, and is the fundamental parameter in our approximation, through which we fix the density. [In SU(3), C^{-3} is the ratio of the probability that there are three quarks on a site to the probability that the site is empty.]

The determinant is easily evaluated by diagonalizing $P_{\vec{x}}$. In SU(2),

$$\det(P_{\vec{x}} + C) = C^2 + C\mathrm{Tr}P_{\vec{x}} + 1, \tag{3}$$

while in SU(3)

$$\det(P_{\vec{x}} + C) = C^3 + C^2\mathrm{Tr}P_{\vec{x}} + C\mathrm{Tr}P_{\vec{x}}^* + 1. \tag{4}$$

In SU(2) this determinant is real and positive, which reflects the fact that quarks and antiquarks are in equivalent representations of SU(2). Unfortunately, studying SU(2) at large baryon density is of limited interest since such baryons would be bosons. In the realistic case of SU(3), we are still left with a complex determinant, albeit a much simpler one, allowing us to generate high statistics.

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In generating gauge configurations, we can update the spatial links with any of the standard algorithms for quenched QCD: Metropolis, heat bath, and/or overrelaxation The temporal links are updated with the Metropolis algorithm, using the magnitude of the determinant plus gauge action as the weight. Thus the parts of the action involved in updating a temporal link U_t are

$$S_{\mathrm{part}} = (2/g^2)\mathrm{ReTr}U_t\tilde{S} + \ln[|\det(U_t\tilde{P}_{\vec{x}} + C)|], \tag{5}$$

where $\tilde{S}$ is the sum of the "staples" and $\tilde{P}_{\vec{x}}$ is the product of all the time direction links at site $\vec{x}$ except for the link being updated. As in conventional quenched QCD, successive Metropolis hits are easy; mot of the work goes into evaluating the staples and $\tilde{P}_{\vec{x}}$, which can be used unchanged in successive hits.

Because of the phase in the determinant, for SU(3) we must estimate expectation values by taking the ratio

$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O}\, e^{i\theta} \rangle_{\|}}{\langle e^{i\theta} \rangle_{\|}}, \tag{6}$$

where θ is the phase of the determinant summed over all spatial points and $\langle\rangle_{\|}$ indicates an expectation value in the ensemble of configurations weighted by the magnitude of the determinant. This method can be applied to the full theory; however, the expectation value of the phase can (and typically does) become very small, so that enormous statistics are required to get meaningful measurements. In our case the phase does become small, however, not prohibitively so on the lattices we study (our smallest value was $\langle e^{i\theta}\rangle_{\|} = 0.02$). Furthermore, as described above, we can produce statistics generously.

The physical quark density is obtained from $\langle n \rangle = (\beta V)^{-1}\partial \ln(Z)/\partial\mu$, where V is the spatial volume and $\beta = an_t$ is the temporal extent of the lattice. Using Eq. (4) this becomes

$$\langle n \rangle = \frac{1}{V}\left\langle \sum_{\vec{x}} \frac{C^2 T_{\vec{x}} + 2CT^*_{\vec{x}} + 3}{C^3 + C^2 T_{\vec{x}} + CT^*_{\vec{x}} + 1} \right\rangle \tag{7}$$

in SU(3), where $T_{\vec{x}} = \mathrm{Tr}(P_{\vec{x}})$. At $C = \infty$ the density is 0; at $C = 0$ the system is saturated with density n_c per site; $C = 1$ represents "half filling" and the density is $n_c/2$.

Note from Eqs. (3) and (4) that $\det(P_i + C)$ is unchanged by the replacement $C \rightarrow 1/C$, where in SU(3) we simultaneously replace $U \rightarrow U^*$. Thus there is a duality relation: The ensemble of configurations generated with coupling C is the same as that generated at coupling $1/C$, and the density obeys $\rho(1/C) = n_c - \rho(C)$. Physically this duality reflects the fact that $n_c - 1$ quarks on a single site behave like an antiquark on that site, so that a small density of holes behaving as antiquarks in a nearly saturated system (at small C) behaves like a system with a small density of quarks (at large C).

We can also find the probability that a site contains zero, one, two, or three quarks. These are just the expectation values of the various powers of C in the determinant, Eqs. (3) and (4). Similarly we may calculate correlation functions $\langle p_n(\vec{x})p_m(\vec{y})\rangle$, the probability for n quarks at site $\vec{x}$ and m quarks at site $\vec{y}$. These correlations may be used to study the clustering properties of baryons in the model as temperature and density are varied.

Since in our approximation the only remaining combination of chemical potential and quark mass is C, the heavy quark condensate $\langle\bar{\Psi}\Psi\rangle = (\beta V)^{-1}\partial\ln(Z)/\partial m$ is trivially related to $\langle n\rangle$. However, we use $\langle\bar{\psi}\psi\rangle$ evaluated for light quarks on the generated lattices as an indicator of chiral symmetry breaking. This is just a probe of the nature of the gauge configurations, rather than a condensate of the actual quarks in the model. It represents the chiral properties of light valence quarks in the presence of a finite density of massive quarks. We also calculate the average Polyakov loop. Not surprisingly, since the heavy quarks are coupled directly to the Polyakov loop, it gets a nonzero expectation value at any C other than zero or infinity. This does not necessarily represent deconfinement, since we have put in a density of quarks which can now shield the test quark represented by the Polyakov loop.

We have run SU(3) simulations on $6^3 \times 2$, $8^3 \times 2$, $10^3 \times 2$, and $6^3 \times 4$ lattices. Typical runs include 500 equilibration sweeps of the lattice and 4000 measuring sweeps, where in each sweep we make two overrelaxation updates of the spatial links and ten Metropolis updates of both spatial and temporal links. The average phase on the $n_t = 2$ lattices ranges from one, at $1/C = 0.0$, to as small as 0.04 (the $10^3 \times 2$ lattice at $6/g^2 = 4.8$ and $1/C = 0.08$). Since all physical observables are obtained from a ratio of expectation values [Eq. (6)] where the numerator and denominator are strongly correlated, we used a jackknife procedure with ten blocks to estimate the errors.

In Fig. 1 we summarize the behavior of the Polyakov loop magnitude ($|P|$) in the T-μ plane, on $N_t = 2$ lattices. This figure shows $\langle|P|\rangle$ as a function of $6/g^2$ (temperature) and $1/C$ (effectively μ). At $1/C = 0$, or zero density, we see the strong first order temperature induced transition at $6/g^2 \approx 5.1$. As the density increases, this transition smooths out. We plot the magnitude of the Polyakov loop rather than its real part because at zero density our simulations average over the Z(3) symmetry of the pure gauge action, which multiplies the Polyakov loop by factors of $e^{2\pi i/3}$.

Figures 2 and 3 show the density and $|P|$ along lines of constant $1/C$ on $6^3 \times 2$, $8^3 \times 2$, and $10^3 \times 2$ lattices. From these plots, and from similar behavior in the light quark $\langle\bar{\psi}\psi\rangle$, it appears that the first order transition at $1/C = 0$ becomes a smooth crossover for any nonzero value of the density. This is surprising, since conventional wisdom says that a nonzero discontinuity at the edge of a phase diagram decreases continuously to zero at some point in the interior of the phase diagram. However, we note that simple functions such as $\tanh[(t - t_c)/h^x]$

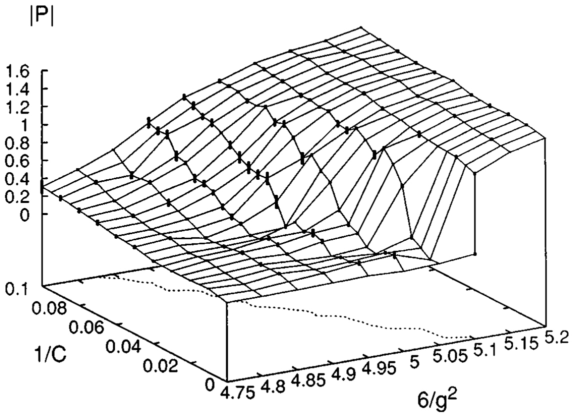

FIG. 1. The magnitude of the Polyakov loop in SU(3) as a function of $6/g^2$ and $1/C$. This plot includes results from $6^3 \times 2$, $8^3 \times 2$, and $10^3 \times 2$ lattices. In the smoother regions of the plot some points were interpolated to produce a regular mesh. The contour line is where $|P| = 0.5$.

have a discontinuity at $h = 0$ but a crossover at any nonzero h. Since we see no systematic dependence of the crossover on the spatial size, except for the expected decrease of the Polyakov loop magnitude on cold lattices, we conclude that this rounding is not a finite size effect. (For a first order transition, we expect the rounding of the jump in observables due to the finite lattice size to diminish quickly with the spatial size, while for $1/C = 0$ we seem to be approaching a smooth curve as the spatial lattice size is increased.) Of course, these Monte Carlo results do not rule out a first order transition line ending at some $1/C$ less than 0.02. Since at 0.02 the density in the dimensionless units of quarks per lattice site is around 0.01 at the crossover (see Fig. 2), and we know of no natural explanation for a change in behavior at a smaller parameter, we think that the simplest explanation for our results is that the transition disappears for any nonzero density.

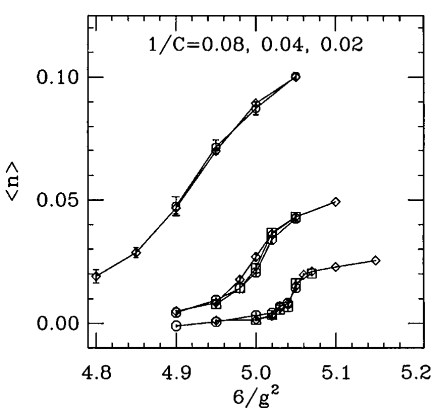

FIG. 2. The density as a function of $6/g^2$ on $n_t = 2$ lattices. The curves are, from left to right, $1/C = 0.08$, 0.04, and 0.02. The $1/C = 0.0$ curve is absent because the density is always zero there. The octagons are $6^3 \times 2$ lattices, the squares $8^3 \times 2$, and the diamonds $10^3 \times 2$. The lines just connect the points for each lattice size.

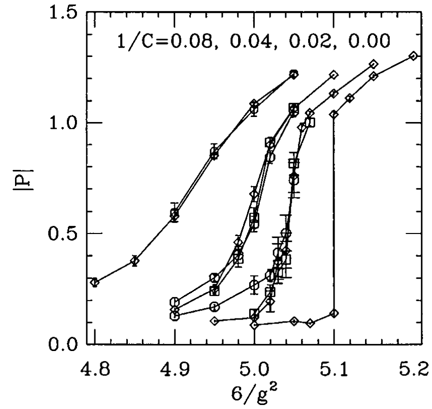

FIG. 3. The magnitude of the Polyakov loop averaged over the lattice as a function of $6/g^2$ on $n_t = 2$ lattices. The curves are, from left to right, $1/C = 0.08$, 0.04, 0.02, and 0.0. The meaning of the symbols is the same as in Fig. 2.

We have also made a series of runs on $6^3 \times 4$ lattices. We began with a series of runs at $6/g^2 = 5.0$. Since the high temperature transition at $1/C = 0$ occurs at $6/g^2 = 5.1$ for $N_t = 2$, this is a fairly cold lattice, with a temperature less than half that for deconfinement at zero density. Figure 4 shows the behavior of the phase $\langle e^{i\theta} \rangle_{\|}$, which in this case gets as small as 0.02. We find that the light quark $\langle \bar{\psi}\psi \rangle$ remains large for any density at this (cold) value of $6/g^2$, as shown in Fig. 4. At $1/C = 1.0$, the density reaches 1.5 quarks per site, where the quarks have the largest effect on the gauge configurations. At this density we find a crossover to restored chiral symmetry at $6/g^2 \approx 5.3$, a significantly lower temperature than the zero density transition, which occurs at $6/g^2 \approx 5.7$. We have also run simulations in SU(2), with similar results.

Clearly there is much more work to be done in the direction of finite density simulations. In this model, we have not yet addressed the question of whether these results scale in the continuum limit nor tried to extract

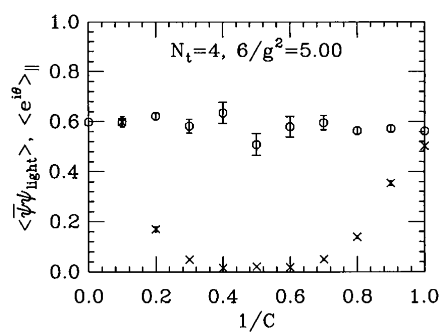

FIG. 4. The light quark $\langle \bar{\psi}\psi \rangle$ (octagons) and $\langle e^{i\theta} \rangle_{\|}$ (crosses) as $1/C$ varies. This is computed at a light quark mass of $am_q = 0.01$. The lattice size is $6^3 \times 4$ and the gauge coupling is $6/g^2 = 5.0$.

physical numbers for various quantities. Perhaps the best hope for progress lies in the use of improved actions, with which it may be possible to approach the continuum limit using lattices that are small enough so that the phase problem is no longer hopeless.

For pure SU(3) (or quenched QCD), the high temperature transition is first order [9], and we expected this behavior to extend into the interior of the T-μ phase diagram. Surprisingly, we find this is apparently not the case. The first order transition appears to become a crossover at nonzero density, becoming quite smooth at relatively low densities.

The mean field analysis of the Potts model in Ref. [7] showed a disappearance of the phase transition at large density. In the *3d* Potts model the first order phase transition persists to some nonzero density (i.e., imaginary magnetic field), with a continuously decreasing discontinuity in the order parameter. The hopping parameter expansion in Ref. [8] showed rather smooth behavior of the Polyakov loop and $\langle\bar{\psi}\psi\rangle$ on the chemical potential.

Does the static approximation have anything to do with real QCD? Certainly the nature of the high temperature transition at zero density depends strongly on the presence of dynamical quarks as is becoming clear from large scale simulations of full QCD [10]. However, it is not *a priori* clear to us that a deconfinement transition or chiral symmetry restoration driven by high density should depend on the quarks moving, or whether the mere presence of the quarks would be enough. In particular, we had not expected to see the zero density first order transition disappear for very small quark densities, or for the signal of chiral symmetry restoration to vanish. This suggests that we might want to reexamine the conventional wisdom that a high density of quarks causes a phase transition similar to that caused by high temperature.

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*Present address: Brookhaven National Lab, Upton, NY 11973.

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