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Instability of Solitons Governed by Quadratic Nonlinearities

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Stability of two-wave solitons supported by resonant parametric interactions in a diffractive (or dispersive) optical quadratic medium is investigated analytically and numerically. It is found that the solitons can become *unstable* when the phase matching between the fundamental and second harmonics is not exactly satisfied. The analytical criterion for the linear instability is presented, and it is revealed that the instability leads to two possible scenarios of the soliton dynamics, either large-amplitude in-phase oscillations of two harmonics or the soliton decay.

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It is well known that propagation of modulated quasi-harmonic waves in dispersive (or diffractive) media of various physical nature is usually governed by the nonlinear Schrödinger (NLS) equation (see, e.g., [1]). However, the NLS equation is not valid near resonances with the second harmonics excited due to a nonlinear (generally *quadratic*) response of a medium. If group velocities of the resonantly interacting harmonics are *essentially different*, dispersion effects are much weaker than the effects of the wave walk-off and nonlinear interaction. In this limit the equations describing three-wave interactions are known to be exactly integrable (see, e.g., [1], and references therein).

When dispersion or diffraction become important, nontrivial effects can be already observed for two interacting waves due to solely parametric interactions. In application to nonlinear optics, this means that the nonlinearity-induced phase shift [2] and self-trapping of light beams [3] can be achieved in the so-called $\chi^{(2)}$ materials due to the cascaded nonlinearities. It has been also shown that the cascaded nonlinearities can support different types of two-wave (spatial or temporal) parametric solitons [3–10], which recently have been observed experimentally in nonlinear planar waveguides [11].

Resonant interaction between the fundamental (w) and second (v) harmonics in a diffractive (or dispersive) $\chi^{(2)}$ medium can be described by the coupled equations for the dimensionless variables [10],

$$
\begin{aligned}
i\frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial x^2} - w + w^* v &= 0, \\
i\sigma\frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial x^2} - \alpha v + \frac{1}{2}w^2 &= 0,
\end{aligned}
\tag{1}
$$

where z is the propagation distance and x stands for the transverse coordinate (spatial solitons) or retarded time (temporal solitons). The parameter σ describes either the ratio of the wave vectors (spatial solitons) or the ratio of the group-velocity dispersions (temporal solitons) of two interacting waves. The parameter α can be presented as $\alpha = 2\sigma - \Delta$, where Δ is proportional to the wave vector mismatch $\Delta k = k_2 - 2k_1$ between the harmonics, and it can also include the walk-off effect [9,10].

The diffraction (dispersion) effects described by the second-order derivatives in Eqs. (1) are crucial for the existence of localized (soliton) solutions. Stationary soliton solutions of Eqs. (1) are presented by real functions $w = w_0(x)$ and $v = v_0(x)$, which are independent of z. For the particular value $\alpha = 1$ these solutions were first found by Karamzin and Sukhorukov [3] and then reproduced again in [6]. Recent numerical analysis [8–10] revealed the existence of the soliton solutions of the system (1) for any value of α and, moreover, it was also shown that

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these solitons can be generated from certain classes of localized initial conditions [9,10].

It is usually believed that two-wave (bright) optical solitons of the model (1) are *stable* [3–9]. For the case $\Delta k = 0$ ($\alpha = 2\sigma$, in our notations), the soliton stability was proven even for the multidimensional case [4], whereas for other cases the stability was demonstrated numerically [6–9].

In this Letter we present, for the first time to our knowledge, a rigorous stability analysis of stationary localized solutions of the model (1) and *reveal instability of parametric solitons* even for the (1 + 1) model, provided the condition of the phase matching is not satisfied.

To analyze the stability of the soliton solutions $w_0(x)$ and $v_0(x)$ with respect to small perturbations, we linearize Eqs. (1) on the soliton background, $w(x,z) = w_0(x) + [W_r(x) + iW_i(x)]e^{\lambda z}$ and $v(x,z) = v_0(x) + [V_r(x) + iV_i(x)]e^{\lambda z}$, and obtain the linear eigenvalue problem

$$\mathbf{L}_+\begin{pmatrix} W_r \\ V_r \end{pmatrix} = \lambda\begin{pmatrix} W_i \\ \sigma V_i \end{pmatrix}, \quad \mathbf{L}_-\begin{pmatrix} W_i \\ V_i \end{pmatrix} = -\lambda\begin{pmatrix} W_r \\ \sigma V_r \end{pmatrix}, \tag{2}$$

where

$$\mathbf{L}_\pm = \begin{pmatrix} d^2/dx^2 - 1 \pm v_0(x), & w_0(x) \\ w_0(x), & d^2/dx^2 - \alpha \end{pmatrix}.$$

It is known [4] that the soliton solutions are stable for $\alpha = 2\sigma$, i.e., when both harmonics are phase matched. However, far from the line $\alpha = 2\sigma$, instability may appear at some critical value $\alpha = \alpha_0(\sigma)$. In the vicinity of the instability threshold $\alpha = \alpha_0(\sigma)$ the growth rate λ is small and we can seek the solutions to Eqs. (2) in the form of asymptotic series in λ,

$$W_{r,i}(x;\lambda) = \sum_{n=0}^{\infty} \lambda^n W_{r,i}^{(n)}(x), \quad V_{r,i}(x;\lambda) = \sum_{n=0}^{\infty} \lambda^n V_{r,i}^{(n)}(x),$$
$$\alpha(\sigma;\lambda) = \sum_{n=0}^{\infty} \lambda^n \alpha_n(\sigma). \tag{3}$$

In the zero-order approximation, Eqs. (2) become decoupled and we find two localized solutions: (i) $W_r^{(0)} = V_r^{(0)} = 0$, $W_i^{(0)} = w_0$, $V_i^{(0)} = 2v_0$, and (ii) $W_r^{(0)} = dw_0/dx$, $V_r^{(0)} = dv_0/dx$, $W_i^{(0)} = V_i^{(0)} = 0$. Solution (i) describes a small shift of the complex phase of the stationary soliton, while solution (ii) describes its coordinate translation. It is clear that both the solutions give the so-called neutral modes of the linear problem (2) at $\lambda = 0$. To describe instability threshold we are interested in the solutions to (2) with *nonzero but small* λ. Such solutions are expected only for special values of the parameters α and σ near the critical curve $\alpha = \alpha_0(\sigma)$. The instability threshold, as well as a general dependence $\alpha(\sigma,\lambda)$, can be found from the corresponding solvability conditions to the linear problem (2). First, we can show that the translational mode (ii) does not become unstable in the whole region of the parameter plane (α,σ). Therefore we consider asymptotic expansions generated by the phase neutral mode (i) and proceed to the next-order approximations of the asymptotic expansions. Then, in the first-order approximation, we find an *explicit analytical solution* for the first corrections,

$$W_r^{(1)} = w_0 + \frac{x}{2}\frac{\partial w_0}{\partial x} + (2\sigma - \alpha)\frac{\partial w_0}{\partial \alpha},$$
$$V_r^{(1)} = v_0 + \frac{x}{2}\frac{\partial v_0}{\partial x} + (2\sigma - \alpha)\frac{\partial v_0}{\partial \alpha}, \tag{4}$$

and $W_i^{(1)} = V_i^{(1)} = 0$.

Substitution of Eqs. (4) into the solvability condition

$$\int_{-\infty}^{+\infty} dx (w_0 W_r + 2\sigma v_0 V_r) = 0 \tag{5}$$

gives us the equation for the instability threshold:

$$4I_1(\alpha,\sigma) \equiv 2(2\sigma - \alpha)\frac{\partial Q(\alpha,\sigma)}{\partial \alpha} + 3Q(\alpha,\sigma) = 0, \tag{6}$$

where $Q(\alpha,\sigma) \equiv \int_{-\infty}^{+\infty} dx\,(w_0^2 + 2\sigma v_0^2)$ is the energy (Menley-Rowe) invariant of Eqs. (1). Equation (6) is a quadratic equation in σ, and its coefficients are expressed only through the stationary soliton solutions which depend on the parameter α. Such soliton solutions have been found for any $\alpha > 0$ in Refs. [8,10] by the numerical shooting technique. Using Eq. (6) and the results from Refs. [8,10], we calculate the instability threshold curve $\alpha = \alpha_0(\sigma)$ and present it in Fig. 1 (solid line). The dashed line shows the curve of the exact resonance $\alpha = 2\sigma$ when both harmonics are phase matched, $k_2 = 2k_1$. When the matching is destroyed and the wave vector of the second harmonic is bigger than the double wave vector

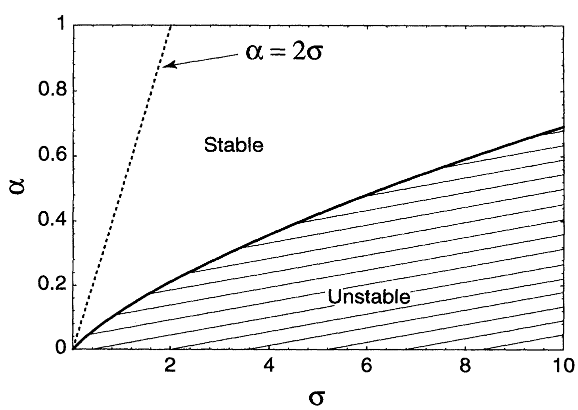

FIG. 1. Region of linear instability of two-wave solitons on the parameter plane (α,σ). The dashed line $\alpha = 2\sigma$ corresponds to the phase-matched interaction between the harmonics at $\Delta k = 0$.

of the first harmonic, $k_2 > 2k_1$, the soliton solutions become unstable for $\alpha < \alpha_0(\sigma)$.

The instability growth rate λ can be estimated analytically from (5), in the second-order approximation, near the instability threshold curve $\alpha = \alpha_0(\sigma)$,

$$\lambda^2 = -\frac{I_1(\alpha, \sigma)}{I_2(\alpha_0, \sigma)} \approx C(\alpha_0, \sigma)(\alpha_0 - \alpha), \quad (7)$$

where $C = I_2^{-1} \partial I_1 / \partial \alpha$ and I_2 is expressed through the higher-order terms of the asymptotic series (3) as follows:

$$I_2(\alpha, \sigma) = \int_{-\infty}^{+\infty} dx \Big(W_r^{(1)} W_i^{(2)} + \sigma V_r^{(1)} V_i^{(2)} \Big).$$

Using a direct analysis similar to that of Ref. [12], we can prove that the integral I_2 is positive for nontrivial localized solutions of Eqs. (2) under the condition $v_0(x) > 0$. This immediately implies that on the plane (α, σ) the instability domain is given by the condition $I_1 < 0$, which is satisfied for $\alpha < \alpha_0(\sigma)$ (see Fig. 1).

For large α, we can seek the solutions to Eqs. (1) in the form of asymptotic series in α^{-1}, when Eqs. (1) transform to a single NLS equation for $w(x, z)$ (see, e.g., [10]). This allows us to find *the exact results* describing the asymptotic behavior of the instability threshold curve (6) and the growth rate (7) for $\alpha \gg 1$: $\sigma \simeq C_\infty \alpha^3$, $\lambda^2 \sim (\sigma - C_\infty \alpha^3)/\alpha^7$, where $C_\infty = 35/128$. It is clear that the stability region expands rapidly for large α, and the growth rate vanishes inside the instability domain.

For the case of spatial optical solitons discussed in Ref. [8] we have $\sigma = 2$, so that, according to the analysis presented above, the two-wave solitons become unstable for $\alpha < \alpha_0 \approx 0.212$ (see Fig. 1). The instability for smaller α's is characterized by the instability growth rate, which can be calculated numerically by investigating the evolution of perturbation eigenmodes. To do this, we use the system of linear equations (2) with the stationary soliton solutions found in [8,10] and calculate the exponentially growing modes which exist for $\alpha < \alpha_0$. The growth rate of the soliton instability for the case $\sigma = 2$ is plotted in Fig. 2. The dashed line represents the result (7) of our analytical theory with $C \approx 0.416$.

The important physical question is *the development of linear instability* in the subsequent dynamics of the two-wave solitons. We have analyzed this problem numerically and found *two different scenarios* of the instability dynamics. As can be seen in Figs. 3(a)–3(c), on one hand the exponential growth of perturbations can be stabilized by nonlinearity, and this leads to periodic amplitude oscillations with a little amount of radiation [less than 2% for the distances presented in Fig. 3(c)]. On the other hand, Figs. 4(a)–4(c) display completely different dynamics: The instability leads to the soliton spreading. This means that diffraction cannot be suppressed by nonlinearity and soliton pulses finally decay into linear diffractive waves.

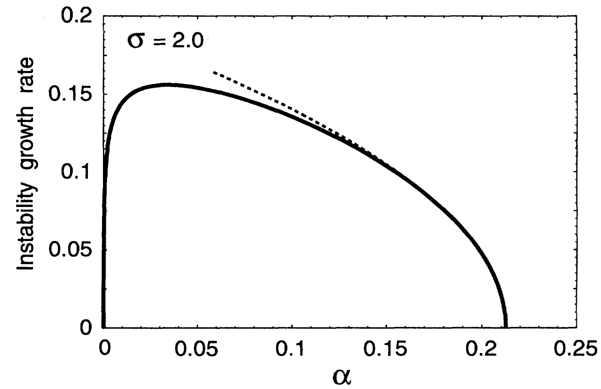

FIG. 2. Growth rate of linear instability at $\sigma = 2$ calculated numerically (solid curve). The analytical asymptotic (7) is shown by a dashed line.

In order to describe analytically these two scenarios we take into account *nonlinear effects* accompanying the development of the linear instability. We select α close to α_0, so that the small parameter ϵ characterizes the deviation $\alpha - \alpha_0 \sim O(\epsilon^2)$. Then, it follows from the linear theory that the growth rate has the order $O(\epsilon)$ and, therefore, the unstable linear perturbations grow on the

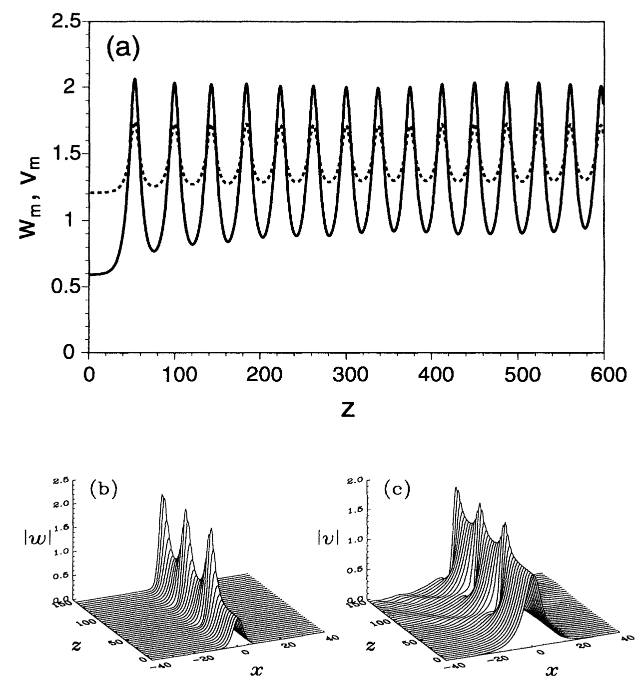

FIG. 3. Characteristic evolution of the unstable solitons for the case $\sigma = 2$, $\alpha = 0.05$, and $\omega(0) > 0$. (a) Evolution of the soliton amplitudes $w_m \equiv |w(0, z)|$ and $v_m \equiv |v(0, z)|$ for the fundamental (solid curve) and second (dashed curve) harmonics. (b), (c) Propagation of the soliton components w and v, respectively.

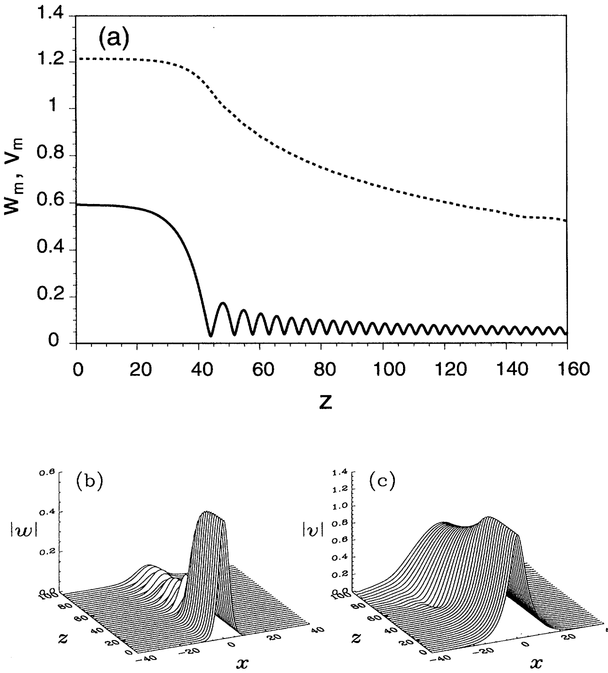

FIG. 4. The same as in Fig. 3 but for $\omega(0) < 0$.

"slow" scale $Z = \epsilon z$. This allows us to introduce the slowly varying complex phase $S = S(Z)$ and look for the perturbed solutions of Eqs. (1) in the form of asymptotic series $w = [w_0(x) + \epsilon^2 \omega W_r^{(1)} + O(\epsilon^3)]e^{i\epsilon S}$, $v = [v_0(x) + \epsilon^2 \omega V_r^{(1)} + O(\epsilon^3)]e^{2i\epsilon S}$, where the functions $W_r^{(1)}$ and $V_r^{(1)}$ are given in (4), and $\omega \equiv dS/dZ$ describes a correction to the wave numbers of the harmonics. Then, the subsequent calculations yield the nonlinear equation for the function ω,

$$\frac{d^2\omega}{dZ^2} - \lambda^2\omega + \gamma\omega^2 = 0, \tag{8}$$

where λ^2 is given by Eq. (7) and $\gamma = 3I_3/2I_2$ characterizes effects due to quadratic nonlinearity. The integral I_3 can be calculated with the help of the stationary soliton solutions [see Eq. (4)] using the explicit formula

$$I_3(\alpha, \sigma) = \int_{-\infty}^{+\infty} dx \Big(W_r^{(1)2} + 2\sigma V_r^{(1)2} - W_r^{(1)2} V_r^{(1)} \Big).$$

We have found numerically that for any $\alpha = \alpha_0(\sigma)$ the integral I_3 is positive and, therefore, $\gamma > 0$.

As follows from Eq. (8), the exponential growth of linear perturbations with $\omega(0) > 0$ is stabilized by nonlinearity leading to oscillations around a novel stable equilibrium state $\omega_0 = \lambda^2/\gamma$. This equilibrium state corresponds to a stationary soliton which can be described also by Eqs. (1) but for a renormalized parameter α lying *inside* the stability domain shown in Fig. 1. Therefore for a slightly increased amplitude of an unstable soliton our analytical model (8) predicts *in-phase* pulsations of the fundamental and second harmonics around a novel stable soliton, and this exactly corresponds to the evolution observed numerically [see Fig. 3(a)].

For $\omega(0) < 0$, according to Eq. (8), such a stabilization is not possible and, therefore, a slightly decreased amplitude of an unstable soliton must gradually decrease. This kind of soliton decay is actually observed numerically as is shown in Figs. 4(a)–4(c). Thus the theory gives two scenarios of the instability of two-wave solitons in diffractive quadratic media, either long-lived almost periodic pulsations or the soliton decay.

The criterion of the soliton instability given by Eq. (6) looks *different* from the well-known criterion of Vakhitov and Kolokolov [13] for the NLS-type equations. In the latter case, the soliton stability is determined by the slope of the dependence of the energy invariant Q vs the soliton propagation constant. However, we have been able to generalize the Vakhitov-Kolokolov criterion to the case of resonant wave interactions in a quadratic nonlinear medium. Indeed, if we make the parameter α (which includes in this case the soliton propagation constant) by an *internal* solution parameter using the following scaling transformation $|\tilde{w}| = a^{-2}|w|(ax, a^2z)$, $|\tilde{v}| = a^{-2}|v|(ax, a^2z)$, where $a^2 = 2\sigma - \alpha > 0$, the energy invariant Q is also changed to be

$$\tilde{Q}(\alpha, \sigma) = \frac{1}{(2\sigma - \alpha)^{3/2}} Q(\alpha, \sigma). \tag{9}$$

Now the condition $\partial\tilde{Q}/\partial\alpha = 0$ *completely coincides* with the criterion (6) derived by the linear stability analysis, because $I_1 = \frac{1}{2}(2\sigma - \alpha)^{5/2}\partial\tilde{Q}/\partial\alpha$. Thus, in such a renormalized form, the soliton stability is determined by the slope of the function $\tilde{Q}(\alpha)$ similar to that in the Vakhitov-Kolokolov criterion, and the two-wave solitons are unstable provided $\partial\tilde{Q}/\partial\alpha < 0$.

In conclusion, for the first time we have found and analyzed the instability of two-wave solitons in diffractive (or dispersive) quadratic media. Our results are extremely important for applications of $\chi^{(2)}$ solitons in all-optical processing and switching, since they give the sufficient criterion for the existence of *stable* parametric solitons ($k_2 \leq 2k_1$). In the opposite case ($k_2 > 2k_1$), the solitons may become unstable. The approach and results obtained can be easily extended to other models of various physical context describing resonant wave interactions governed by quadratic nonlinearities.

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