A New Phase of Tethered Membranes: Tubules

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We show that fluctuating tethered membranes with *any* intrinsic anisotropy unavoidably exhibit a new phase between the previously predicted "flat" and "crumpled" phases, in high spatial dimensions d where the crumpled phase exists. In this new "tubule" phase, the membrane is crumpled in one direction but extended nearly straight in the other. Its average thickness is $R_G \sim L^{\nu_t}$ with L the intrinsic size of the membrane. This phase is more likely to persist down to $d = 3$ than the crumpled phase. In Flory theory, the universal exponent $\nu_t = 3/4$, which we conjecture is an exact result. We study the elasticity and fluctuations of the tubule state, and the transitions into it.

Tethered membranes are of great interest in large part because their behavior is much richer than that of polymers, their one-dimensional analog. Specifically, polymerized membranes have been predicted [1] to undergo a "crumpling" transition between the "crumpled" and long-ranged orientationally ordered "flat" phases. This apparent violation of the Mermin-Wagner theorem is made possible by "anomalous elasticity" [1,2]: Thermal fluctuations infinitely enhance the membrane's effective bending rigidity κ, stabilizing the orientational order against these very fluctuations.

Most past theoretical work [3] has been restricted to *isotropic* membranes. Here we consider *intrinsically anisotropic* membranes and find that this seemingly innocuous modification has profound and surprising consequences: An entire new and heretofore unanticipated phase of the membrane, which we call the "tubule" phase, ubiquitously intervenes between the crumpled and "flat" phases (see Fig. 1). Only in the special case of perfectly isotropic membranes, which follow a path like P_2, is a direct crumpled-to-flat (CF) transition possible. Generic paths like P_1 have *two* phase transitions, crumpled-to-tubule (CT) and tubule-to-flat (TF), which we are currently studying in an ϵ-expansion [4].

There are a number of possible experimental realizations of anisotropic membranes. One is polymerized membranes with in-plane tilt order. *Fluid* membranes with such order have already been found [5]; it should be possible to polymerize these without destroying the tilt order. Secondly, membranes could be fabricated by crosslinking DNA molecules trapped in a fluid membrane [5]. Performing crosslinking in an applied electric field would align the DNA and "freeze in" the anisotropy induced by the electric field, which could then be removed.

Simulations could be done on, e.g., triangular or rectangular nets of balls and springs with all of the spring constants in one direction different, by a factor of order 2 or so, from those in the other direction. Equivalently, one could have different bond *lengths* in the two directions, or use second nearest neighbor springs of different strengths to create different bend stiffnesses in the two directions. *Any* such modification *whatsoever* will lead, upon renormalization, to a membrane with *all* of the anisotropic terms we consider here, and, hence, will fall into the universality class of our model.

The defining property of the tubule phase is that the membrane is crumpled in one direction (y), but "flat" in the other. Its average shape is a long, thin cylinder of length $R_y = L_y^0 \times O(1)$ and radius $R_G(L_\perp) \ll L_\perp$, where L_y and $L_\perp$ are the intrinsic dimensions of the membrane. The tubule radius R_G, and its undulations h_{rms} transverse to the y axis, obey the scaling laws

$$R_G(L_\perp, L_y) \propto L_\perp^{\nu_t}, \tag{1}$$

$$h_{\mathrm{rms}}(L_\perp, L_y) = L_y^\zeta f_h(\Lambda L_y/(\Lambda L_\perp)^z), \tag{2}$$
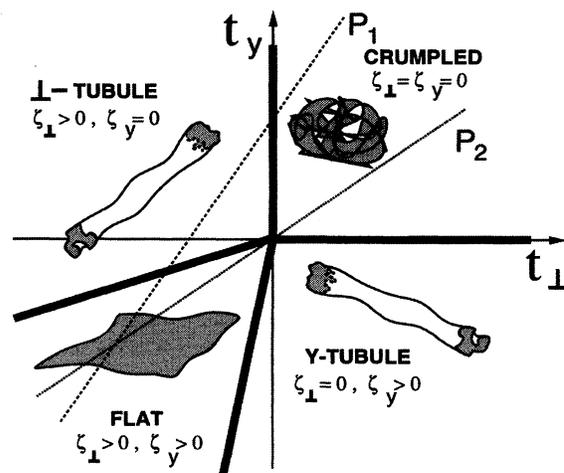

FIG. 1. Phase diagram for anisotropic tethered membranes showing the flat, crumpled, and new tubule phases.

where the universal exponents $z = (1 + \nu_t)/2, \zeta = (1 - \nu_t)/[2(1 + \nu_t)]$ are < 1, Λ is an ultraviolet cutoff, $f_h(u) \to$ const for $u \to 0$ and $f_h(u) \propto u^{3/2-\zeta}$, for $u \to \infty$.

For general spatial dimension d, Flory theory treatment of the self-avoidance (SA) predicts $\nu_t = 3/(d + 1)$, suggesting that the tubule phase should be stable down to the lower critical dimension $d_{lc} = 2$, where $\nu = 1$, and therefore should exist in 3D, predicting $\nu_t = \frac{3}{4}$, which implies that $z = \frac{7}{8}$, and $\zeta = \frac{1}{14}$. However, the analogous Flory result for the crumpled phase $\nu = 4/(d + 2)$ has more recently been called into question. Numerical simulations [6] find no crumpled phase below $d = 4$. An uncontrolled Gaussian approximation [7,8] supports this finding, predicting $\nu = 4/d$, which suggests that $d_{lc} = 4$ for the crumpled phase. Both this and the numerical values of ν for $d > 4$ agree well with the simulations.

The same approximation for the tubule phase [4] gives $\nu = 7/(3d - 5)$, which suggests that $d_{lc} = 4$ for the tubule phase as well. However, despite its success for the crumpled phase, this Gaussian approximation is known to be far from trustworthy. For example, it predicts $\nu = 2/d$ for linear polymers, which not only is less accurate than the Flory result $\nu = 3/(d + 2)$, but also incorrectly predicts $d_{lc} = 2$ for polymers, when, in fact, it is well known that $d_{lc} = 1$ in that case.

Whether the Gaussian approximation is any more reliable for our tubule phase remains an open question. One could argue that a slice perpendicular to the y axis through our tubule looks like a SA random walk in two dimensions, for which the Flory result of $\nu = 3/4$ is known to be *exact,* while the Gaussian approximation $\nu = 1$ is clearly wrong. Whether or not this analogy holds, it *is* clear that SA, though a relevant perturbation, is much less important for tubules than for the crumpled phase, since points on the membrane widely separated in the y direction never bump into each other in the tubule phase, while they do in the crumpled phase. So it seems quite plausible, the Gaussian approximation

notwithstanding, that the tubule phase *is* stable in $d = 3$. Furthermore, the suppression of the crumpled phase by SA makes the possibility of the new tubule phase even more interesting and important. Whether the tubule phase *does* survive in $d = 3$ can be determined only by simulations and experiments on anisotropic membranes, both of which we hope our work stimulates.

In the following discussions, numerical estimates for the values of the exponents will be obtained from the Flory estimate, $\nu_t = \frac{3}{4}$ in 3D; these numbers should be taken with a grain of salt, due to the uncertainties just discussed about the validity of the Flory theory.

Equation (2) implies that $h_{rms}(L) \propto L^{1-\nu_t} \approx L^{1/4}$, for a square membrane with $L_\perp \sim L_y \equiv L$, using $\lim_{L\to\infty} u \equiv \Lambda L_y/(\Lambda L_\perp)^z \to \infty$. In the "linear polymer" limit $L_y \gg L_\perp$, $h_{rms} \propto L_y^{3/2}/L_\perp^{z(3/2-\zeta)} = L_y^{3/2}/L_\perp^{\nu_t+1/2} \approx L_y^{3/2}/L_\perp^{5/4}$. Defining L_P to be the value of L_y at which $h_{rms} = L_y$, we obtain the orientational persistence length $L_P \propto L_\perp^{2\nu_t+1} \approx L_\perp^{5/2}$. For any roughly square membrane, L_y is much less than L_P, hence the tubule phase is stable against thermal fluctuations as $L \to \infty$.

Like the flat phase [1,2,9], the tubule phase exhibits anomalous elasticity; however, as discussed above, SA is a strongly relevant perturbation in this new phase. The tubule, swelled by the SA interaction, acts for $L_y \gg L_y^c \equiv \Lambda^{-1}(\Lambda L_\perp)^z$ like a polymer with bending rigidity $\kappa_p(L_\perp) \propto L_\perp R_G^2 \propto L_\perp^{1+2\nu_t} \approx L_\perp^{5/2}$.

Our model for anisotropic membranes is a generalization of the isotropic model [10]. We characterize the configuration of the membrane by the position $\vec{r}(\mathbf{x})$ in the d-dimensional embedding space of the point in the membrane labeled by a D-dimensional internal coordinate $\mathbf{x}$. In the physical case, $D = 2$ and $d = 3$. The Landau free energy F is an expansion in the local tangent vectors $\partial_\alpha \vec{r}(\mathbf{x})$, keeping only the leading terms consistent with global translation and rotation invariance:

$$F[\vec{r}(\mathbf{x})] = \frac{1}{2} \int d^{D-1}x_\perp \, dy \left[\kappa_\perp(\partial_\perp^2 \vec{r})^2 + \kappa_y(\partial_y^2\vec{r})^2 + \kappa_{\perp y}\partial_y^2\vec{r} \cdot \partial_\perp^2 \vec{r} + t_\perp(\partial_\alpha^\perp \vec{r})^2 + t_y(\partial_y\vec{r})^2 \right.$$
$$+ \frac{u_{\perp\perp}}{2}(\partial_\alpha^\perp\vec{r} \cdot \partial_\beta^\perp\vec{r})^2 + \frac{u_{yy}}{2}(\partial_y\vec{r} \cdot \partial_y\vec{r})^2 + u_{\perp y}(\partial_\alpha^\perp\vec{r} \cdot \partial_y\vec{r})^2 + \frac{v_{\perp\perp}}{2}(\partial_\alpha^\perp\vec{r} \cdot \partial_\alpha^\perp\vec{r})^2$$
$$\left. + v_{\perp y}(\partial_\alpha^\perp\vec{r})^2 (\partial_y\vec{r})^2 \right] + \frac{b}{2} \int d^Dx \int d^Dx' \delta^{(d)}(\vec{r}(\mathbf{x}) - \vec{r}(\mathbf{x}')), \tag{3}$$

where the κ's, t's, u's, and v's are elastic constants, and b is the SA interaction strength. The first three (κ) terms in F represent the anistotropic bending energy of the membrane. The elastic constants $t_\perp$ and t_y are >0 at high temperatures and <0 at low temperatures. When both are positive, the membrane crumples. When either is negative, the membrane extends in the associated direction, and the u and v quartic terms are then needed for stability. Equation (3) reduces to the model for isotropic membranes [10] when $t_\perp = t_y, \kappa_{\perp\perp} = \kappa_y, \kappa_{\perp y} = 0, u_{yy} = 4(\tilde{v} + u), u_{\perp\perp} = u_{\perp y} = 4u$, and $v_{\perp\perp} = v_{\perp y} = 4\tilde{v}$.

In mean-field theory, we seek a configuration $\vec{r}(\mathbf{x})$ that minimizes the free energy Eq. (3). Since the curvature energies $\kappa_\perp(\partial_\perp^2 \vec{r})^2$ and $\kappa_y(\partial_y^2\vec{r})^2$ vanish when $\vec{r}(\mathbf{x})$ is linear in $\mathbf{x}$, we seek these minima by inserting the ansatz $\vec{r}_0(\mathbf{x}) = (\zeta_\perp \mathbf{x}_\perp, \zeta_y y, 0, 0, \ldots, 0)$ into Eq. (3). For now neglecting the SA interaction,

$$F = \frac{1}{2} L_\perp^{D-1} L_y \left[t_y\zeta_y^2 + t_\perp(D - 1)\zeta_\perp^2 \right.$$
$$+ \frac{1}{2} u'_{\perp\perp}(D - 1)^2\zeta_\perp^4 + \frac{1}{2} u_{yy}\zeta_y^4$$
$$\left. + v_{\perp y}(D - 1)\zeta_\perp^2\zeta_y^2 \right], \tag{4}$$

where $u'_{\perp\perp} \equiv v_{\perp\perp} + u_{\perp\perp}/(D-1)$. Minimizing F over $\zeta_\perp$ and ζ_y yields two possible phase diagram topologies. For $u'_{\perp\perp}u_{yy} > v^2_{\perp y}$, we obtain Fig. 1. Both $\zeta_\perp$ and ζ_y vanish for $t_\perp, t_y > 0$. This is the crumpled phase: the entire membrane, in mean-field theory, collapses into the origin $\vec{r}(\mathbf{x}) = 0$ for all $\mathbf{x}$. In our new y-tubule phase, characterized by $\zeta_\perp = 0$ and $\zeta_y = \sqrt{|t_y|/u_{yy}} > 0$, the membrane is extended in the y direction but crumpled in the $\perp$ directions. The $\perp$-tubule phase is the analogous phase with the y and $\perp$ directions reversed. The tubule-flat boundary slopes are $u_{yy}/v_{\perp y}$ and $v_{\perp y}/u'_{\perp\perp}$, respectively. In the flat phase, both $\zeta_\perp$ and $\zeta_y \neq 0$. For $u'_{\perp\perp}u_{yy} < v^2_{\perp y}$, the flat phase disappears and is replaced by a direct first-order transition from $\perp$-tubule to y-tubule along the locus $t_y = (v_{\perp y}/u'_{\perp\perp})t_\perp$.

The flat and crumpled phases of anisotropic membranes in Fig. 1 are in the same universality class [11] as those of isotropic membranes [1]. In the crumpled phase, $t_\perp, t_y > 0$ and all other local terms in Eq. (3) are irrelevant at long wavelengths. A change of variables $\mathbf{x}_\perp = \mathbf{x}'\sqrt{t_\perp/t_y}$ makes the remaining energy isotropic.

We now consider the effects of fluctuations, ignoring SA (i.e., the "phantom" membrane). Consider the y-tubule phase. To treat fluctuations, we perturb around the mean-field solution by writing $\vec{r}(\mathbf{x}) = (\zeta_y y + u(\mathbf{x}), \vec{h}(\mathbf{x}))$, where $\vec{h}(\mathbf{x})$ is a $(d-1)$-component vector orthogonal to y. Inserting the above expression for $\vec{r}$ into Eq. (3) and keeping only relevant terms gives $F_{\text{tot}} = F_{\text{MFT}} + F_{\text{el}}$, where $F_{\text{MFT}} = \frac{1}{2} L^{D-1}_\perp L_y[t_y\zeta^2_y + \frac{1}{4}u_{yy}\zeta^4_y]$,

$$F_{\text{el}} = \frac{1}{2}\int d^{D-1}x_\perp\,dy\left\{\kappa(\partial^2_y\vec{h})^2 + t(\partial^\perp_\alpha\vec{h})^2 + g_\perp(\partial^\perp_\alpha u)^2 \right.$$
$$\left. + g_y\left[\partial_y u + \frac{1}{2}(\partial_y\vec{h})^2\right]^2\right\}, \quad (5)$$

$\kappa \equiv \kappa_y$, $t \equiv t_\perp + v_{\perp y}\zeta^2_y$, $g_y \equiv u_{yy}\zeta^2_y/2$, and $g_\perp \equiv t + u_{\perp y}\zeta^2_y$.

Note that the ratios of the coefficients of the quadratic $(\partial_y u)^2$ and the anharmonic $\partial_y u(\partial_y\vec{h})^2$ and $(\partial_y\vec{h})^4$ terms in F_{el} must be exactly 4:4:1, since they must appear together as a result of expanding the rotationally invariant combination $[\partial_y u + \frac{1}{2}(\partial_y\vec{h})^2]^2$. This ratio allows us to calculate exactly the long-wavelength anomalous elasticity of "phantom" tubules, as we will show in a moment.

The propagators can be read off from Eq. (5), giving $\langle h_i(\mathbf{q})h_j(-\mathbf{q})\rangle = k_B T\delta^\perp_{ij}G_h(\mathbf{q})$, $\langle u(\mathbf{q})u(-\mathbf{q})\rangle = k_B T G_u(\mathbf{q})$, where $G^{-1}_h(\mathbf{q}) = tq^2_\perp + \kappa q^4_y$, $G^{-1}_u(\mathbf{q}) = g_\perp q^2_\perp + g_y q^2_y$, and $\delta^\perp_{ij}$ is a Kronecker delta when both indices i and $j \neq y$, and is zero if either i or $j = y$. The rms fluctuations in the harmonic approximation are

$$\langle|\vec{h}(\mathbf{x})|^2\rangle \propto \int_{L_\perp}\frac{d^{D-1}q_\perp\,dq_y}{(2\pi)^D}\frac{1}{tq^2_\perp + \kappa q^4_y} \propto L^{5/2-D}_\perp, \quad (6)$$

clearly revealing that the upper critical dimension is $D_{uc} = 5/2$. Below D_{uc}, we expect anomalous elasticity. However, this anomaly is not manifested in the fluctuations of $\vec{h}$ alone. We can see this by integrating out the phonons u exactly. The only remaining anharmonicity in the effective elastic free energy for $\vec{h}$ alone is

$$F_{anh}[\vec{h}] = \frac{1}{4}\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}[\vec{h}(\mathbf{k}_1)\cdot\vec{h}(\mathbf{k}_2)][\vec{h}(\mathbf{k}_3)\cdot\vec{h}(\mathbf{k}_4)]$$
$$\times(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_3\cdot\mathbf{k}_4)V_h(\mathbf{q}), \quad (7)$$

where $\mathbf{q} = \mathbf{k}_1 + \mathbf{k}_2$ and $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 = 0$. The effective vertex is $V_h(\mathbf{q}) = g_y g_\perp q^2_\perp/(g_y q^2_y + g_\perp q^2_\perp)$, and is irrelevant for $D > 3/2$, as can be seen by simple power counting. Thus in $D = 2$, the elastic constants t, $g_\perp$, and κ_y are finite and nonzero as $q_y \to 0$.

However, g_y is driven to zero as $q_y \to 0$. In a self-consistent one-loop perturbative calculation, similar to that successfully used to compute the anomalous elasticity in the flat phase [9], we find

$$g_y(\mathbf{q}) = g^0_y - \int\frac{(k_B T)^3 g^2_y(\mathbf{q})p^2_y(p_y - q_y)^2\,d^{D-1}p_\perp dp_y/(2\pi)^D}{[tp^2_\perp + \kappa(\mathbf{p})p^4_y][t|\mathbf{p}_\perp - \mathbf{q}_\perp|^2 + \kappa(|\mathbf{p} - \mathbf{q}|)(p_y - q_y)^4]}, \quad (8)$$

where g^0_y is the "bare" value of g_y. The above argument shows that $\kappa(\mathbf{p})$ can be replaced by a constant, since the $\vec{h}$ elasticity is not anomalous. This self-consistent equation can be solved by the ansatz $g_y(\mathbf{q}) = q^{\eta_u}_y f_g(q_y/q^z_\perp)$ with $z = \frac{1}{2}$, $\eta_u = 5 - 2D$, which we have verified works to all orders in perturbation theory.

We now compute the phantom tubule diameter R_G and transverse wandering roughness h_{rms}, defined by $R^2_G \equiv \langle|\vec{h}(\mathbf{L}_\perp, y) - \vec{h}(0_\perp, y)|^2\rangle$, $h^2_{\text{rms}} \equiv \langle|\vec{h}(\mathbf{x}_\perp, L_y) - \vec{h}(\mathbf{x}_\perp, 0)|^2\rangle$. Because h_{rms} and R_G receive large contributions from $\mathbf{q}_\perp = 0$ and $q_y = 0$ zero modes, respectively, R_G and h_{rms}, surprisingly, scale

in different ways with the membrane dimensions $L_\perp$ and L_y. Taking into account the zero modes, we calculate h_{rms} and R_G by equipartition, and find the forms Eqs. (1) and (2) with $z = 1/2$, $\nu_t = (5 - 2D)/4$, and $\zeta = 2\nu_t$. For a nearly square membrane $L_\perp \sim L_y \sim L \to \infty$, for which $\Delta L_y \gg (\Delta L_\perp)^z$, we obtain $h_{\text{rms}} \propto L^3_y/L^{(D-1)/2}_\perp \propto L^{2-D/2}$. Thus for a $D = 2$ phantom tubule, $h_{\text{rms}} \propto L$. Unlike the flat phase, no $\ln(L/a)$ correction arises, so the $(D = 2)$ phantom tubule is just marginally stable, but with wild transverse undulations. These are greatly suppressed by SA, to the effects of which we now turn.

We begin by estimating the radius of gyration using Flory theory. Specializing henceforth to the physical case $D = 2$ for simplicity, we estimate the SA energy E_{SA} as $E_{SA} \approx bV\rho^2$, where the volume V in the embedding space occupied by the tubule is $V \approx R_G^{d-1}L_y$, and the density ρ in the embedding space of the tubule is $\rho \approx M/V \propto L_\perp L_y/V$, where M is the mass of the membrane. Putting these together gives $E_{SA} \propto L_y L_\perp^2/R_G^{d-1}$.

Inserting our earlier, phantom membrane result $R_G \propto L_\perp^{1/4}$ (for $D = 2$) and taking $L_\perp \propto L_y^2$, as required by anisotropic scaling, we find that $E_{SA} \propto L_y^{5-(d-1)/2}$, which goes to infinity as $L_y \to \infty$ for $d < 11$. Thus SA is strongly relevant, and changes the long wavelength behavior of the membrane, for $d < 11$.

We can calculate R_G for $d < 11$ by combining the above estimate of E_{SA} with a similar scaling estimate of the elastic energy yielding

$$E_{FL} = \left[t_y \zeta_y^2 + u_{yy}\zeta_y^4 + t\left(\frac{R_G}{L_\perp}\right)^2 \right] L_\perp L_y + b\frac{L_y L_\perp^2}{\zeta_y R_G^{d-1}}. \quad (9)$$

Minimizing this over R_G, we obtain $R_G(L_\perp) \propto L_\perp^{\nu_t}$, with $\nu_t = 3/(d + 1)$. For the physical case $d = 3$, this gives $\nu_t = 3/4$. Since a slice through the tubule traces out a crumpled polymer embedded in 2D, we conjecture that $\nu_t = 3/4$ is an *exact* result for the tubule thickness, since it is for 2D polymers. For a square membrane $L_y \sim L_\perp$, it is straightforward to argue that the $q_y = 0$ zero modes do not contribute to R_G, and $L_\perp$ is the dominant infrared cutoff. Hence Eq. (1) gives the correct radius of gyration.

This highly nontrivial *ground state* for the SA tubule is *not* modified by thermal fluctuations. That is, even at $T > 0$, the variation of R_G with $L_\perp$ is completely dominated by the SA energy, i.e., determined by a $T = 0$ fixed point. This can be seen by evaluating the elastic (or SA) energy with $R_G(L_\perp)$ given by this SA ground state. For an $L \times L$ membrane, this energy $E_{el} \propto L^\theta$, $\theta = 6/(d + 1) = \frac{3}{2}$ (for $d = 3$), is $\gg k_B T$ as $L \to \infty$.

Using a generalization of Landau's derivation of shell theory we now calculate anomalous elasticity in the presence of SA. Bending the tubule with radius of curvature $R_c \gg R_G$ induces a strain $\epsilon \sim R_G/R_c$, which costs an additional elastic energy density $g_y(L_y, L_\perp)\epsilon^2 = g_y(L_y, L_\perp)[R_G(L_y)/R_c]^2$. Interpreting this additional energy as an effective bending energy density $\kappa_y(L_\perp, L_y)/R_c^2$ leads to the *effective* bend modulus $\kappa_y(L_\perp, L_y)$,

$$\kappa_y(L_\perp, L_y) \sim g_y(L_\perp, L_y)R_G(L_\perp, L_y)^2. \quad (10)$$

Inserting $\kappa_y(L_\perp, L_y) = L_y^\eta f_\kappa(\Lambda L_y/(\Lambda L_\perp)^z)$, $g_y(L_\perp, L_y) = L_y^{-\eta_u}f_g(\Lambda L_y/(\Lambda L_\perp)^z)$, $R_G(L_\perp, L_y) = L_\perp^{\nu_t}f_R(\Lambda L_y/(\Lambda L_\perp)^z)$ into the above expression, we obtain a relation between the exponents $2\nu_t = z(\eta + \eta_u)$, which is satisfied by our earlier results for the phantom tubule.

The physical SA tubule at $T = 0$ is absolutely straight (i.e., $\zeta_y = 1$ *exactly*). This implies that the tubule stretching elastic constant $g_y = g_y^0$, its bare value, since there are

neither fluctuations nor SA effects to renormalize it (in contrast to κ). Therefore $\eta_u = 0$ at $T = 0$; Eq. (10) then implies that κ_0 for the $T = 0$ SA tubule is already anomalous, and given by $\kappa_0(\mathbf{q}) = q_y^{-2\nu_t/z}f_\kappa(q_y\Lambda^z/\Lambda q_\perp^z)$.

The effective free energy describing thermal fluctuations about this nontrivial, SA ground state is

$$F_{eff} \approx \frac{1}{2} \int \frac{dq_\perp dq_y}{(2\pi)^2} \left[\kappa_0(\mathbf{q})q_y^4 + tq_\perp^2 \right] |\vec{h}(\mathbf{q})|^2. \quad (11)$$

Balancing the κq_y^4 term in this expression with the $tq_\perp^2$ term gives the anisotropy of scaling exponent z, defined by $q_y \propto q_\perp^z$. We thereby obtain $z = 1/(2 - \nu_t/z)$, which gives $z = (1 + \nu_t)/2$. Now calculating the fluctuation corrections to g_y from Eq. (8) using the wave-vector dependent κ found above, we find that the integral in Eq. (8) converges in the infrared and g_y is finite as $\mathbf{q} \to 0$ provided $2 > \frac{5}{2}(1 - \nu_t)$. If ν_t is anywhere near its Flory value $\nu_t = 3/4$ in $D = 2, d = 3$ this condition is clearly satisfied. Thus unlike the phantom, the SA tubule has $\eta_u = 0$, even at $T > 0$. Using this fact in Eq. (10) and taking the Flory expression for ν_t in $d = 3$, we obtain, $z = (1 + \nu_t)/2 = \frac{7}{8}$, $\eta = 4\nu_t/(1 + \nu_t) = \frac{12}{7}$, $\eta_u = 0$.

Using Eq. (11) to compute the transverse tubule undulations h_{rms}, we find $h_{rms} = L_y^\zeta f_h(\Lambda L_y/(\Lambda L_\perp)^z)$, where $\zeta = \frac{3}{2} - (1 + 2\nu_t)/(2z)$, and $f_h(x) \approx x^{(1+2\nu_t)/(2z)}$ and $f_h(x) \approx$ const for $x \to \infty$ and $x \to 0$, respectively. For a roughly square $L \times L$ membrane, we are always in the $x \to \infty$ limit of the crossover function $f_h(x)$, which implies that the rms transverse undulations of the tubule are given by $h_{rms} \sim L^{3/2-(1+2\nu_t)/2} \sim L^{1/4}$. Similar arguments applied to R_G yield Eq. (1).

We [4] are currently investigating the effects of SA on the tubule phase in $d = d_{uc}^{SA} - \epsilon$. The scaling theory of the tubule-to-flat and crumpled-to-tubule transitions, along with a Flory theory for these transitions, will be presented in a future publication [4].

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