New Value of the α^3 Electron Anomalous Magnetic Moment

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Highly accurate numerical evaluation of the sixth-order term of the electron anomalous magnetic moment a_e has detected an error in the analytic value of a fourth-order infrared-divergent integral, which is needed to obtain the best estimate of the α^3 term as a combination of analytic values of 67 Feynman diagrams and numerical values of 5 diagrams for which no analytic results are known. Correction of this error leads to a small but significant revision of the α^3 term. As a consequence the fine structure constant α determined from theory and experiment of a_e is reduced by 55×10^{-9}.

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The anomalous magnetic moment of the electron a_e is one of the simplest quantities precisely calculable from first principles. Furthermore, it has been measured very accurately [1]:

$$a_e(\text{expt}) = 1\,159\,652\,188.4(4.3) \times 10^{-12}. \tag{1}$$

Thus it plays a crucial role in testing the validity of QED, or, more generally, the standard model. An even more rigorous test will become feasible when the forthcoming experiments are completed [2]. To make such a test meaningful, however, it is necessary to improve the sixth-order (α^3) and eighth-order (α^4) terms of a_e.

This paper reports the result of a new numerical evaluation of the α^3 term. All 72 Feynman diagrams contributing to the α^3 term have been evaluated numerically [3], and all but 5 are now known analytically [4–6]. The new calculation confirms the analytic results to a much higher degree, and reduces the uncertainty in the remaining 5 diagrams by an order of magnitude. More importantly, however, improved precision of the calculation has led to the discovery of a small error in the analytic value of a fourth-order infrared-divergent (IR-divergent) integral, which is needed to obtain the best estimate of the α^3 term as a "hybrid" of analytic values of 67 Feynman diagrams and numerical values of the remaining 5 diagrams. Correction of this error leads to a small (0.44%) but significant revision of the α^3 term. This has the effect of reducing the fine structure constant α determined from theory and experiment of a_e by 55×10^{-9}.

The QED part of the contribution can be expressed as

$$a_e^{\text{QED}} = A_1 + A_2(m_e/m_\mu) + A_2(m_e/m_\tau) + A_3(m_e/m_\mu, m_e/m_\tau),$$

where m_e, m_μ, and m_τ are the masses of the electron, muon, and tau, respectively. A_2 and A_3 as well as the contributions of the hadronic and weak interactions are very small and known with sufficient accuracy. They add up to [3]

$$\Delta a_e = 4.46(20) \times 10^{-12}. \tag{2}$$

Thus far the first four coefficients in the perturbation expansion of the mass-independent A_1 term

$$A_1 = A_1^{(2)}(\alpha/\pi) + A_1^{(4)}(\alpha/\pi)^2 + A_1^{(6)}(\alpha/\pi)^3 + A_1^{(8)}(\alpha/\pi)^4 + \cdots, \tag{3}$$

representing 1, 7, 72, and 891 Feynman diagrams, respectively, have been evaluated. Previously reported values of these coefficients are [3]

$$\begin{aligned} A_1^{(2)} &= 0.5, \\ A_1^{(4)} &= -0.328\,478\,965\ldots, \\ A_1^{(6)} &= 1.176\,11(42), \\ A_1^{(8)} &= -1.434(138). \end{aligned} \tag{4}$$

$A_1^{(2)}$ and $A_1^{(4)}$ are known analytically. The value of $A_1^{(8)}$ is determined by purely numerical means using the Monte Carlo integration routine VEGAS [7]. On the other hand, the value of $A_1^{(6)}$ quoted in (4) is a "hybrid" obtained by combining the analytic results for 51 diagrams and the best numerical values of 21 diagrams for which no exact values were available. (Of the latter, 16 have since been evaluated analytically [5].)

22 of the diagrams contributing to $A_1^{(6)}$ contain closed electron loops of vacuum-polarization or light-light scattering type. Numerical and analytic values of these contributions agree very well [3,5,6]. The remaining 50 diagrams are represented by the eight self-energy-like diagrams $6A, \ldots, 6H$ of Fig. 1. The method developed in [8] expresses the renormalized magnetic moments $a_{6A}, \ldots, a_{6H}$ [omitting the factor $(\alpha/\pi)^3$ for simplicity] as follows:

$$a_{6A} = \Delta M_{6A} - 2\tilde{B}_2 \Delta M_{4b} + (2I_{4s} + \tilde{B}_2^2 - 2\tilde{L}_2\tilde{B}_2)M_2, \tag{5}$$

$$\begin{aligned} a_{6B} = \Delta M_{6B} &+ (\tilde{L}_2 - \tilde{B}_2)\Delta M_{4b} \\ &+ \Delta\delta m_{4b}(M_{2^*}[I] - M_{2^*}) \\ &+ [-\Delta B_{4b} + 2I_{4s} + 2I_{4l} \\ &\quad + (\tilde{B}_2 - \tilde{L}_2)^2]M_2, \end{aligned} \tag{6}$$

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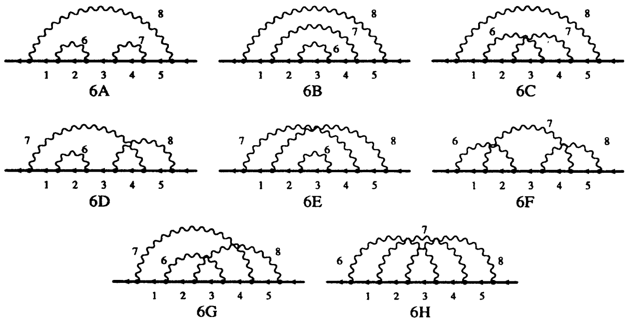

FIG. 1. Sixth-order self-energy-like diagrams representing 50 vertex diagrams with three virtual photons. 5 vertex diagrams are generated by insertion of an external magnetic field in the electron line of each diagram. Diagrams related to 6D and 6G by time reversal are not shown.

$$a_{6C} = \Delta M_{6C} + \tilde{L}_2 \Delta M_{4a} - 2\tilde{L}_2 \Delta M_{4b} + \Delta\delta m_{4a}(M_{2^*}[I] - M_{2^*}) + (-\Delta B_{4a} + I_{4x} + 2I_{4c} - 2\tilde{L}_2^2 + 2\tilde{L}_2\tilde{B}_2)M_2 , \tag{7}$$

$$a_{6D} = 2\Delta M_{6D} - 2\tilde{B}_2 \Delta M_{4a} - 2\tilde{L}_2 \Delta M_{4b} + 2(-\Delta L_{4s} - I_{4s} + I_{4c} + 2\tilde{L}_2\tilde{B}_2 - \tilde{L}_2^2)M_2 , \tag{8}$$

$$a_{6E} = \Delta M_{6E} - \tilde{B}_2 \Delta M_{4a} - 2(\Delta L_{4s} + I_{4s} - \tilde{L}_2\tilde{B}_2)M_2 + I_{4x}M_2 , \tag{9}$$

$$a_{6F} = \Delta M_{6F} - 2\tilde{L}_2 \Delta M_{4a} - 2(\Delta L_{4c} + I_{4c})M_2 + 3\tilde{L}_2^2 M_2 , \tag{10}$$

$$a_{6G} = 2\Delta M_{6G} - 2\tilde{L}_2 \Delta M_{4a} - 2(\Delta L_{4l} + \Delta L_{4c} + I_{4l} + I_{4c})M_2 + 2\tilde{L}_2^2 M_2 , \tag{11}$$

$$a_{6H} = \Delta M_{6H} - 2(\Delta L_{4x} + I_{4x})M_2 . \tag{12}$$

Here $\Delta M_{6A}, \ldots, \Delta M_{6H}$ are the UV- and IR-finite parts of the sixth-order integrals defined in [3]. ΔM_{4a}, ΔM_{4b}, $\Delta\delta m_{4a}$, $\Delta\delta m_{4b}$, $\Delta L_{4s}, \ldots,$ and M_2, M_{2^*}, $M_{2^*}[I]$ are finite quantities of fourth and second order. Fourth-order terms I_{4x}, I_{4c}, I_{4l}, I_{4s} and second-order terms $\tilde{B}_2$, $\tilde{L}_2$ are IR divergent.

Combining (5)–(12) one obtains

$$a_e^{(6)}(\text{Fig. 1}) = \sum_{\alpha=A}^{H} \eta_\alpha \Delta M_{6\alpha} - 3\Delta B_2 \Delta M^{(4)} + \Delta\delta m^{(4)}(M_{2^*}[I] - M_{2^*}) - [\Delta B^{(4)} + 2\Delta L^{(4)} - 2(\Delta B_2)^2]M_2 , \tag{13}$$

where $\eta_\alpha = 2$ for $\alpha = D, G$, $\eta_\alpha = 1$ otherwise, and

$$\Delta B_2 = \tilde{B}_2 + \tilde{L}_2 ,$$
$$\Delta M^{(4)} = \Delta M_{4a} + \Delta M_{4b} ,$$
$$\Delta\delta m^{(4)} = \Delta\delta m_{4a} + \Delta\delta m_{4b} ,$$
$$\Delta B^{(4)} = \Delta B_{4a} + \Delta B_{4b} ,$$
$$\Delta L^{(4)} = \Delta L_{4x} + 2\Delta L_{4c} + \Delta L_{4l} + 2\Delta L_{4s} .$$

Note that I_{4x}, I_{4c}, I_{4l}, and I_{4s} are absent in (13). Thus there is no need to compute them explicitly. Values of the second-order quantities are $M_2 = 0.5$, $M_{2^*} = 1.0$, $M_{2^*}[I] = -1.0$, and $\Delta B_2 = 0.75$.

The result of the pre-1990 evaluation of (13) is

$$a_e^{(6)}(\text{Fig. 1; 1990}) = 0.905\,1(86) . \tag{14}$$

The details of this calculation are given in Table X and Table XI of Ref. [3]. An analytic value corresponding to (14) is not available since diagram 6*H* is not yet known analytically.

Nevertheless, it was possible to obtain a value more accurate than (14) by combining the analytic and semianalytic results known for the diagrams 6*A* to 6*G* [4] with the purely numerical result for the diagram 6*H*. For this purpose, as is seen from (12), one has to know the IR-divergent integral I_{4x}, which was evaluated analytically by Sapirstein [9]. Using his result

$$I_{4x} = \Delta I_{4x} - \ln\lambda ,$$
$$\Delta I_{4x} = -2.504\,839 , \tag{15}$$

where λ is the IR cutoff mass in units of electron mass, one obtains the hybrid value

$$a_e^{(6)}(\text{Fig. 1; 1990}h) = 0.899\,87(42) . \tag{16}$$

The difference 0.005 2(86) between (14) and (16) was well within the uncertainty of the former, apparently justifying (15). Thus $A_1^{(6)}$ in (4) was chosen in Ref. [3] as the sum of (16) and the contribution of 22 diagrams containing closed electron loops. Since then analytic evaluation of the latter was completed, giving [5]

$$0.276\,262\ldots . \tag{17}$$

This modifies $A_1^{(6)}$ of (4) only slightly.

In order to compare the theory with the measurement it is necessary to know the value of the fine structure constant α. Currently there are three measurements of α whose precision exceeds 0.1 part in a million (0.1 ppm). They are based on the quantum Hall effect [10], the ac Josephson effect [11], and the measurement of the ratio of Planck's constant h and the neutron mass m_n [12]:

$$\alpha^{-1}(\text{q. Hall}) = 137.035\,997\,9(32) \quad (0.024\text{ ppm}) ,$$
$$\alpha^{-1}(\text{acJ}) = 137.035\,977\,0(77) \quad (0.056\text{ ppm}) ,$$
$$\alpha^{-1}(h/m_n) = 137.036\,010\,82(524) \quad (0.039\text{ ppm}) . \tag{18}$$

Substitution of these α's in (3) leads to the theoretical predictions

$$a_e(\text{q. Hall}) = 1\,159\,652\,140.4(5.3)(4.1)(27.1) \times 10^{-12},$$
$$a_e(\text{acJ}) = 1\,159\,652\,317.0(5.3)(4.1)(65.3) \times 10^{-12},$$
$$a_e(h/m_n) = 1\,159\,652\,027.7(5.3)(4.1)(44.4) \times 10^{-12},$$

where the three errors on each line are due to the uncertainties in $A_1^{(6)}$, $A_1^{(8)}$, and α used in the evaluation. These values are about -1.7, $+2.0$, and -3.6 standard deviations away from the experiment (1).

If one wants to explain the difference between $a_e(\text{expt})$ and $a_e(h/m_n)$ in terms of the eighth-order effect, one needs $A_1^{(8)} \sim +4$. Although the real error of $A_1^{(8)}$ might be larger than that of (4) by a factor of 2 or even 3, which is not inconceivable because of insufficient Monte Carlo sampling of the integrand in the quoted calculation, the probability that the result of VEGAS integration is off by as much as 40 standard deviations is infinitesimal.

Another avenue to be explored is the sixth-order term $A_1^{(6)}$. In this case we must increase $A_1^{(6)}$ by 0.013 or 1.1% to explain $a_e(\text{expt}) - a_e(h/m_n)$. This is about 30 times larger than the uncertainty of $A_1^{(6)}$ quoted in (4), which again looks highly unlikely. Note, however, that $A_1^{(6)}$ in (4) is a hybrid value. Its error may arise not only from numerical integration of the sixth-order integrals but also from the fourth-order IR-divergent integral I_{4x}.

To examine this question I have evaluated $a_e^{(6)}$ (Fig. 1) numerically once again with much (more than 100 times) higher statistics. The results of the new calculation are given in Table I. Substituting these results and the new values of auxiliary integrals listed in Table II in Eq. (13), one obtains

$$a_e^{(6)}(\text{Fig. 1; 1995}) = 0.904\,882(347)\,. \qquad (19)$$

This is in good agreement with (14).

TABLE I. The contribution of sixth-order diagrams of Fig. 1 to the electron anomalous magnetic moment. $\eta_\alpha = 1(2)$ for time-reversal symmetric (asymmetric) diagrams. All integrals are evaluated in double precision except in a small domain of $6A$, where the integral is evaluated in quadruple precision (using 10^7 function calls, 60 iterations) as is indicated by the second numbers in the row for $6A$; similarly for $6G$.

Diagram	$\eta_\alpha \Delta M_{6\alpha}$	Function calls per iteration (in units of 10^8)	No. of iterations
$6A$	$-1.354\,698(89)$	40, 0.1	60, 60
$6B$	$3.018\,838(152)$	40	60
$6C$	$-0.335\,204(142)$	40	60
$6D$	$0.928\,666(138)$	80	62
$6E$	$1.198\,785(121)$	20	60
$6F$	$0.753\,486(107)$	40	65
$6G$	$2.467\,476(143)$	10, 1	60, 60
$6H$	$-2.206\,436(39)$	200	68

TABLE II. Auxiliary integrals. ΔM_{4a} and ΔM_{4b} are known exactly.

Integral	Value	Function calls per iteration (in units of 10^8)	No. of iterations
ΔM_{4a}	$0.218\,333\,12\ldots$		
ΔM_{4b}	$-0.187\,5$		
$\Delta\delta m_{4a}$	$-0.301\,600(14)$	20	65
$\Delta\delta m_{4b}$	$2.207\,939(15)$	20	64
ΔB_{4a}	$-0.039\,811(15)$	10	63
ΔB_{4b}	$-0.397\,282(15)$	10	64
ΔL_{4x}	$-0.481\,852(8)$	10	55
ΔL_{4c}	$0.003\,378(6)$	10	56
ΔL_{4l}	$0.124\,814(3)$	10	60
ΔL_{4s}	$0.407\,653(4)$	10	56

On the other hand, (19) disagrees with the hybrid result (16) by more than 9 standard deviations. To obtain the hybrid $A_1^{(6)}$ one must know the fourth-order integral I_{4x}. Unfortunately, this integral, evaluated analytically by Sapirstein [9], has not been checked independently. In order to provide such a check, I have evaluated the IR-finite part ΔI_{4x} or I_{4x} in several ways. I have also evaluated ΔI_{4c}, ΔI_{4l}, and ΔI_{4s}, where [13]

$$I_{4c} = \Delta I_{4c} + \frac{5}{4}\ln\lambda + \frac{1}{2}(\ln\lambda)^2,$$
$$I_{4l} = \Delta I_{4l} - \frac{1}{4}\ln\lambda - \frac{1}{2}(\ln\lambda)^2,$$
$$I_{4s} = \Delta I_{4s} - \frac{1}{2}\ln\lambda - \frac{1}{2}(\ln\lambda)^2,$$

since they are needed to check the analytic results of $a_{6A}, \ldots, a_{6G}$, individually. [As was mentioned above, there is no need to know them in order to calculate $a_e^{(6)}$ (Fig. 1) from Eq. (13). It is for this reason that I have not evaluated them in the past.] Instead of direct analytic evaluation, I have evaluated them in two nonanalytic ways. One is by a straightforward numerical integration of exact Feynman-parametric integrals. Column 2 of Table III lists the values of ΔI_{4x}, ΔI_{4c}, ΔI_{4l}, and ΔI_{4s} thus obtained. Another is by comparison of the analytic

TABLE III. Various evaluations of ΔI_{4x}, ΔI_{4c}, ΔI_{4l}, and ΔI_{4s}. All integrals listed in column 2 have been evaluated using 10^9 function calls per iteration and about 60 iterations.

Integral	Numerical integration	Combination of formulas	Formulas used
ΔI_{4x}	$-2.509\,977(31)$	$-2.509\,935(301)$	(5), (9)
		$-2.510\,118(361)$	(7), (10)
ΔI_{4c}	$0.996\,035(28)$	$0.995\,886(108)$	(10)
ΔI_{4l}	$2.277\,918(23)$	$2.278\,051(179)$	(5), (6)
		$2.277\,941(179)$	(10), (11)
ΔI_{4s}	$0.262\,707(27)$	$0.262\,749(89)$	(5)

values of $a_{6A}, \ldots, a_{6G}$ with the corresponding numerical integration results by means of the equations (5)–(11). Column 3 lists the results obtained in this way. Column 4 lists the formulas used in deriving column 3.

The results in Table III clearly disagree with ΔI_{4x} of (15). Informed of these results Sapirstein has evaluated it analytically once again and obtained the new value [14]

$$\Delta I_{4x} = \frac{39}{8} + (\ln 2 - 5)\zeta(2) - \frac{1}{4}\zeta(3) = -2.510\,003\,149\ldots, \quad (20)$$

where $\zeta(n)$ is the Riemann ζ function. This is in excellent agreement with the values given in Table III. Knowing the result (20) one can improve the hybrid value of $a_e^{(6)}$ (Fig. 1), namely, the combination of the analytic results for $a_{6A}, \ldots, a_{6G}$ [4,5] and the numerically obtained value of a_{6H}. The new value is

$$a_e^{(6)}(\text{Fig. } 1; 1995h) = 0.904\,997(40).$$

This is in good agreement with (19).

Our calculation shows that the analytic and numerical results of $a_{6A}, \ldots, a_{6G}$ agree, in the order listed in Table I, within about 0.46, 1.10, 1.40, 1.13, 0.07, 1.34, and 0.85 times the uncertainties of numerical integration. The overall agreement between analytic and numerical results is excellent, enhancing confidence in both results.

As for the numerical precision of a_{6H}, the consistency of all previous evaluations on various computers and with various sizes of sampling statistics indicates that the error given in Table I can be trusted. Combining this with the rest of the sixth-order correction given in (17), one finds

$$A_1^{(6)} = 1.181\,259(40).$$

Note that this uncertainty comes entirely from a_{6H}.

Using this and a still tentative value of $A_1^{(8)}$ given in [15], and including Δa_e of (2), one obtains the new value

$$a_e(\text{q. Hall}) = 1\,159\,652\,201.4(0.5)(2.1)(27.1) \times 10^{-12}, \quad (21)$$

where the errors are due to the uncertainties in $A_1^{(6)}$, $A_1^{(8)}$, and α(q. Hall), respectively. This is in agreement with experiment within the uncertainty of theory. On the other hand, the revised values of a_e(acJ) and $a_e(h/m_n)$ are about +2.9 and −2.3 standard deviations away from experiment. Currently numerical work is in progress to improve $A_1^{(8)}$, which will reduce the second error of (21).

At present it is not possible to test QED more rigorously because of the uncertainties in the measurements of α listed in (18). A more sensible way to test QED is to compare these α's with the α determined from the theory and experiment of a_e:

$$\alpha^{-1}(a_e) = 137.035\,999\,44(57) \quad (0.0042\text{ ppm}).$$

The uncertainty comes mostly from experiment [1]. Note that this α is 5 to 13 times more precise than those listed in (18).

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