Ferromagnetism in Hubbard Models

Hal Tasaki*
Department of Physics, Gakushuin University, Mejiro, Toshima-ku, Tokyo 171, Japan
(Received 7 April 1995)

We present the first rigorous examples of nonsingular Hubbard models which exhibit ferromagnetism at zero temperature. The models are defined in arbitrary dimensions, and are characterized by finite-range hoppings, dispersive bands, and finite on-site Coulomb interaction U. The picture, which goes back to Heisenberg, that sufficiently large Coulomb interaction can revert Pauli paramagnetism into ferromagnetism has been confirmed in concrete examples.

PACS numbers: 75.10.Lp

The origin of ferromagnetism has been a mystery in physical science for quite a long time [1]. It was Heisenberg [2] who first realized that ferromagnetism is intrinsically a quantum many-body effect, and proposed the scenario that spin-independent Coulomb interaction and the Pauli exclusion principle generate "exchange interaction" between electronic spins. One of the motivations to study the so-called Hubbard model has been to establish and understand the generation of ferromagnetism in simplified situations [3,4]. Unfortunately, rigorous examples of ferromagnetism (or ferrimagnetism) in the Hubbard models have been limited to singular models which have infinitely large Coulomb interaction (Nagaoka-Thouless ferromagnetism [5]), or in which magnetization is supported by a dispersionless band (Lieb's ferrimagnetism [6], and flatband ferromagnetism due to Mielke [7] and the present author [8]). In [9,10], local stability of ferromagnetism in a generic family of Hubbard models with nearly flat bands was proved.

In the present Letter, we treat a class of Hubbard models in arbitrary dimensions, which are nonsingular in the sense that they have finite-range hoppings, dispersive (single-electron) bands, and finite Coulomb interaction U. We prove that the models exhibit ferromagnetism in their ground states provided that U is sufficiently large. We recall that Hubbard models with dispersive bands (like ours) exhibit Pauli paramagnetism when $U = 0$, and remain nonferromagnetic for sufficiently small U. The appearance of ferromagnetism is a purely nonperturbative phenomenon.

As far as we know, this is the first time that the existence of ferromagnetism has been established in nonsingular itinerant electron systems. We stress that our examples finally provide the definite affirmative answer to the long-standing fundamental problem: *whether spin-independent Coulomb interaction can be the origin of ferromagnetism in itinerant electron systems* [11]. See [8,10,12] for further discussions of ferromagnetism in the Hubbard models.

In order to simplify the discussion, we describe our results in one-dimensional models. We discuss models in higher dimensions at the end of the Letter. Let N be an arbitrary integer, and let Λ be the set of integers x with $|x| \le N$. We identify $x = -N$ and $x = N$ to regard Λ as a periodic chain with $2N$ sites. We denote by $\mathcal{E}$ and $\mathcal{O}$ the subsets of Λ, consisting of even and odd sites (integers), respectively. As usual, we denote by $c^{\dagger}_{x,\sigma}$, $c_{x,\sigma}$, and $n_{x,\sigma}$ the creation, the annihilation, and the number operators, respectively, for an electron at site $x \in \Lambda$ with spin $\sigma = \uparrow, \downarrow$.

We consider the standard Hubbard Hamiltonian

$$H = \sum_{\substack{x,y\in\Lambda \\ \sigma=\uparrow,\downarrow}} t_{x,y}\, c^{\dagger}_{x,\sigma} c_{y,\sigma} + U \sum_{x\in\Lambda} n_{x,\uparrow} n_{x,\downarrow}, \qquad (1)$$

where $t_{x,x+1} = t_{x+1,x} = t'$ for any $x \in \Lambda$, $t_{x,x+2} = t_{x+2,x} = t$ if $x \in \mathcal{E}$, $t_{x,x+2} = t_{x+2,x} = -s$ if $x \in \mathcal{O}$, and $t_{x,x} = V$ if $x \in \mathcal{O}$. The remaining elements of $t_{x,y}$ vanish. See Fig. 1. Here s, t, and U are positive parameters [13]. The parameters t' and V are determined by s, t, and another positive parameter λ as $t' = \lambda(s + t)$ and $V = (\lambda^2 - 2)(s + t)$. Our main theorem applies to the case $\lambda = \sqrt{2}$, where we have $V = 0$. We consider the Hilbert space with N electrons in the system. This corresponds to the quarter filling of all the bands, or the half filling of the lower band.

If we consider the single-electron problem corresponding to the Hamiltonian (1), we find that the model has two bands with dispersion relations $\varepsilon_1(k) = -2s\cos 2k - 2(s + t)$ and $\varepsilon_2(k) = 2t\cos 2k + \lambda^2(s + t)$ with $|k| \le \pi/2$. Note that both bands have perfect cosine dispersions, which is a special feature of the present model [14]. There is an energy gap $\lambda^2(s + t)$ between the two bands.

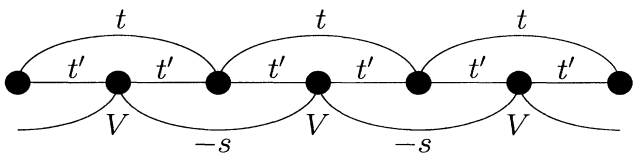

FIG. 1 The one-dimensional lattice and the hopping matrix elements. There are hoppings to nearest and next-nearest neighbors, and on-site potential.

 0031-9007/95/75(25)/4678(4)$06.00

For $\alpha = 1, 2, 3$, we define the total spin operators by $S_{\rm tot}^{(\alpha)} = \sum_{x\in\Lambda} \sum_{\sigma,\tau=\uparrow,\downarrow} c_{x,\sigma}^{\dagger} (p^{(\alpha)}/2)_{\sigma,\tau} c_{x,\tau}$, where $p^{(\alpha)}$ are the Pauli matrices, and denote the eigenvalues of $(\mathbf{S}_{\rm tot})^2 = \sum_{\alpha=1}^{3} (S_{\rm tot}^{(\alpha)})^2$ as $S_{\rm tot}(S_{\rm tot} + 1)$. The maximum possible value of $S_{\rm tot}$ is $S_{\rm max} \equiv N/2$.

Let $b_{k,\sigma}^{\dagger}$ be the creation operator corresponding to the single-electron eigenstate with the energy $\varepsilon_1(k)$. Let $\Phi_{\rm vac}$ be the state with no electrons. The state $\Phi_{\rm ferro} = (\prod_k b_{k,\uparrow}^{\dagger})\Phi_{\rm vac}$ (where the lower band is fully filled by up-spin electrons) has the lowest energy among the states with $S_{\rm tot} = S_{\rm max}$. It is easy to observe that $\Phi_{\rm ferro}$ is an eigenstate of H with the energy $E_0 = -2(s + t)N$. A simple variational calculation shows that $\Phi_{\rm ferro}$ cannot be a ground state of H (and hence the true ground state has $S_{\rm tot} < S_{\rm max}$) if $U < 4s$. The main result of the present Letter is the following.

Theorem I.—Suppose $\lambda > \lambda_c = [(2 + \sqrt{5})^{1/2} - 2]^{1/2} \simeq 0.241$. If t/s and U/s are sufficiently large, the ground states of the Hamiltonian (1) have $S_{\rm tot} = S_{\rm max}$, and are nondegenerate apart from the $(2S_{\rm tot} + 1)$-fold spin degeneracy. $\Phi_{\rm ferro}$ is one of the ground states. How large the parameters should be can be determined by diagonalizing a Hubbard model on a five-site chain (see Fig. 2).

The present models reduce to the flatband models studied in [8] if we set $s = 0$. Therefore we can regard theorem I as a confirmation in special cases of the previous conjecture [8–10,15] that the flatband ferromagnetism is stable under perturbations. Moreover, the strong result about the spin-wave excitation proved in [9,10] also applies to the preset models.

Theorem II.—Suppose that the model parameters satisfy $\lambda \geq \lambda_3$, $s/t \leq \rho_0$, and $K_2\lambda t \geq U \geq A_3\lambda^2 s$, where λ_3, ρ_0, K_2, and A_3 are positive constants that appear in [10]. Then the spin-wave excitation energy $E_{\rm SW}(k)$ [i.e., the lowest energy among the states with $(N - 1)$ up-spin electrons and one down-spin electron, and with crystal momentum k] of the Hubbard model (1) satisfies

$$F_2 \frac{4U}{\lambda^4} (\sin k)^2 \leq E_{\rm SW}(k) - E_0 \leq F_1 \frac{4U}{\lambda^4} (\sin k)^2, \quad (2)$$

where E_0 is the ground state energy, and F_1 and F_2 are constants such that $F_1 \simeq F_2 \simeq 1$ if $\lambda \gg 1$, $\lambda s \ll t$, and $U \gg \lambda^2 s$. (See [10] for details and a proof.)

In the parameter region where both theorems I and II are applicable, we have an ideal situation where the global stability of ferromagnetism as well as the appearance of "healthy" low-lying excited states are rigorously established. We have derived rigorously a ferromagnetic system with (effective) exchange interaction $J \simeq 2U/\lambda^4$, starting from the Hubbard models for itinerant electrons.

It is quite likely that the present models represent (Mott-Hubbard) insulators. We expect that the same models with smaller electron numbers describe metallic ferromagnetism [16], but have no rigorous results in this direction (except for those in the flatband models [8]).

Proof of theorem I.— For $x \in \Lambda$ and $\sigma = \uparrow, \downarrow$, we define $a_{x,\sigma} = \lambda c_{x,\sigma} - (-1)^x (c_{x-1,\sigma} + c_{x+1,\sigma})$, which correspond to the strictly localized basis states used in [8–10]. The anticommutator $\{a_{x,\sigma}^{\dagger}, a_{y,\sigma}\}$ is $\lambda^2 + 2$ if $x = y$, 1 if $|x - y| = 2$, and vanishing otherwise. By using these operators, Hamiltonian (1) can be written in a compact manner as

$$H = (\lambda^2 s - 2t)N - s \sum_{\substack{x\in\mathcal{E} \\ \sigma=\uparrow,\downarrow}} a_{x,\sigma}^{\dagger} a_{x,\sigma} + t \sum_{\substack{x\in\mathcal{O} \\ \sigma=\uparrow,\downarrow}} a_{x,\sigma}^{\dagger} a_{x,\sigma} + U \sum_{x\in\Lambda} n_{x,\uparrow} n_{x,\downarrow} \quad (3)$$

in the sector with N electrons. We further rewrite it as $H = (\lambda^2 s - 2t)N + \sum_{x\in\mathcal{E}} h_x$ with the local Hamiltonian defined as

$$h_x = -s \sum_{\sigma=\uparrow,\downarrow} a_{x,\sigma}^{\dagger} a_{x,\sigma} + U n_{x,\uparrow} n_{x,\downarrow} + \sum_{r=\pm 1} \left(\frac{t}{2} \sum_{\sigma=\uparrow,\downarrow} a_{x+r,\sigma}^{\dagger} a_{x+r,\sigma} + \frac{U}{2} n_{x+r,\uparrow} n_{x+r,\downarrow} \right). \quad (4)$$

Since $[h_x, h_{x+2}] \neq 0$, it is impossible to diagonalize all h_x simultaneously.

Lemma.—Suppose $\lambda > \lambda_c$, and t/s and U/s are sufficiently large. Then the minimum eigenvalue of h_x (regarded as an operator on the whole Hilbert space) is $-(\lambda^2 + 2)s$. In any of the corresponding eigenstates, there are one, two, or three electrons in the sublattice $\{x - 2, x - 1, x, x + 1, x + 2\}$, and these electrons are coupled *ferromagnetically*. Any eigenstate Φ with the eigenvalue $-(\lambda^2 + 2)s$ can be written in the form,

$$\Phi = a_{x,\uparrow}^{\dagger} \Phi_1 + a_{x,\downarrow}^{\dagger} \Phi_2, \quad (5)$$

with some states Φ_1 and Φ_2, and satisfies

$$c_{x+1,\downarrow} c_{x+1,\uparrow} \Phi = 0. \quad (6)$$

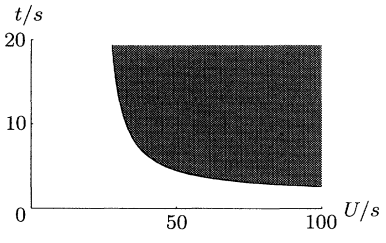

FIG. 2 When $\lambda = \sqrt{2}$, theorem I is applicable for t/s and U/s in the shaded region. The existence of ferromagnetism is established, for example, if $t \geq 4.5\,s$ when $U = 50\,s$, or $t \geq 2.6\,s$ when $U = 100\,s$. Though the plot was obtained from a numerical calculation in a five-site Hubbard model, our theorem guarantees that the model with *arbitrary* lattice size exhibits ferromagnetism.

We prove the lemma in the next part. In what follows, we assume that the model parameters satisfy the conditions in the lemma. The basic strategy of the proof of theorem I is to extend the local ferromagnetism found above into a global ferromagnetism. Special characters of the present model make such an extension possible.

The lemma implies $h_x \geq -(\lambda^2+2)s$, and hence $H \geq (\lambda^2 s - 2t)N - N(\lambda^2+2)s = -2(s+t)N = E_0$. This proves that $\Phi_{\rm ferro}$ (which has the eigenenergy E_0) is a ground state.

To show the uniqueness of the ground states, we assume Φ is a ground state, i.e., $H\Phi = E_0\Phi$. Then we have $h_x\Phi = -(\lambda^2+2)s\,\Phi$ for each $x \in \mathcal{L}$, and Φ is characterized by the lemma. We note that the collection of states $(\prod_{x\in A} a^\dagger_{x,\uparrow})(\prod_{x\in B} a^\dagger_{x,\downarrow})\Phi_{\rm vac}$ with arbitrary subsets $A, B \subset \Lambda$ such that $|A| + |B| = N$ forms a (complete) basis of the N-electron Hilbert space. Imagine that we expand Φ using this basis. Since (5) holds for any $x \in \mathcal{L}$, Φ must be written in the form

$$\Phi = \sum_{\tilde\sigma} \varphi(\tilde\sigma)\left(\prod_{x\in\mathcal{L}} a^\dagger_{x,\sigma(x)}\right)\Phi_{\rm vac}\,, \tag{7}$$

where $\tilde\sigma = [\sigma(x)]_{x\in\mathcal{L}}$ is a spin configuration with $\sigma(x) = \uparrow, \downarrow$, and $\varphi(\tilde\sigma)$ is a coefficient. Unlike in the flat-band models [8], a state of the form (7) is not necessarily an eigenstate of the hopping part of H.

By examining how $c_{x+1,\downarrow}c_{x+1,\uparrow}$ acts on (7), the condition (6) reduces to

$$\varphi(\tilde\sigma) = \varphi(\tilde\sigma_{x,x+2}) \quad \text{for any } \tilde\sigma\,, \tag{8}$$

where $\tilde\sigma_{x,x+2}$ is the spin configuration obtained by switching $\sigma(x)$ and $\sigma(x+2)$ in $\tilde\sigma$. Since (8) holds for any $x \in \mathcal{L}$, we find that $\varphi(\tilde\sigma) = \varphi(\tilde\tau)$ whenever $\sum_{x\in\mathcal{L}}\sigma(x) = \sum_{x\in\mathcal{L}}\tau(x)$. Since $\Phi_{\rm ferro}$ is written as $\Phi_{\rm ferro} = {\rm const}(\prod_{x\in\mathcal{L}} a^\dagger_{x,\uparrow})\Phi_{\rm vac}$, this means that Φ can be written in the form $\Phi = \sum_{M=0}^{N}\alpha_M (S^-_{\rm tot})^M \Phi_{\rm ferro}$, where the spin lowering operator is $S^-_{\rm tot} = \sum_{x\in\Lambda} c^\dagger_{x,\downarrow}c_{x,\uparrow}$. This proves that $\Phi_{\rm ferro}$ and its SU(2) rotations are the only ground states of H.

Proof of lemma.—Because of the translation invariance, it suffices to prove the lemma for $x = 0$. We first diagonalize the hopping part of h_0 (obtained by setting $U = 0$). We express a single-electron state supported on the sublattice $\Lambda_0 = \{-2, -1, 0, 1, 2\}$ as a five-dimensional vector $\varphi = (\varphi_{-2}, \varphi_{-1}, \varphi_0, \varphi_1, \varphi_2)$. The normalized eigenstates are $\varphi^{(0)} = (\lambda^2+2)^{-1/2}(0, -1, \lambda, -1, 0)$ with the eigenvalue $\varepsilon_0 = -(\lambda^2+2)s$, $\varphi^{(1)} = \{2(\lambda^2+1)\}^{-1/2}(\lambda, -1, 0, 1, -\lambda)$ with $\varepsilon_1 = 0$, $\varphi^{(2)} = \{2(\lambda^2+2)(\lambda^2+3)\}^{-1/2}[-(\lambda^2+2), \lambda, 2, \lambda, -(\lambda^2+2)]$ with $\varepsilon_2 = 0$, and two more with $\varepsilon_3 = (\lambda^2+1)t/2$ and $\varepsilon_4 = (\lambda^2+3)t/2$. We denote the corresponding creation operators as $d^\dagger_{i,\sigma} = \sum_{x\in\Lambda_0}\varphi^{(i)}_x c^\dagger_{x,\sigma}$. It is crucial to note that $a^\dagger_{0,\sigma} = (\lambda^2+2)^{1/2}d^\dagger_{0,\sigma}$.

Since the local Hamiltonian h_0 conserves the number of electrons in Λ_0, we can examine its minimum eigenvalue in each sector with a fixed number of electrons in Λ_0. When there are no electrons in Λ_0, the only possible eigenvalue of h_0 is $0 > -(\lambda^2+2)s$. Let f_n and e_n be the minimum eigenvalues of h_0 in the sectors with n electrons in Λ_0 with the total spin (of the n electrons) $S^{(n)}_{\rm tot} = n/2$ and $S^{(n)}_{\rm tot} < n/2$, respectively.

Noting that $f_n = \sum_{m=0}^{n-1}\varepsilon_m$, we find $f_1 = f_2 = f_3 = -(\lambda^2+2)s$ and $f_n > -(\lambda^2+2)s$ for $n \geq 4$. Since the corresponding ferromagnetic eigenstates are $(\prod_{m=0}^{n-1} d^\dagger_{m,\uparrow})\tilde\Phi$ (where $\tilde\Phi$ is an arbitrary state with no electrons in Λ_0) or their SU(2) rotations, they are written in the desired form (5), and satisfy (6). Therefore, in order to prove the lemma, it suffices to show that

$$e_n > -(\lambda^2+2)s \quad \text{for any } n = 2, 3, \ldots, 8\,. \tag{9}$$

Since condition (9) only involves eigenvalues of a finite system, it can be checked by numerically diagonalizing finite dimensional matrices for given values of λ, s, t, and U. We can thus construct a *computer aided proof* that our Hubbard model exhibits ferromagnetism. Figure 2 summarizes the result of a preliminary analysis in this direction.

Let us prove (9) in a range of parameters without using computers. Let $e_2^{\rm sym}$ ($e_2^{\rm as}$) be the minimum eigenvalue of h_0 in the sector with two electrons in Λ_0 forming spin-singlet states which is symmetric (antisymmetric) under the spatial reflection $x \to -x$. Let us evaluate $e_2^{\rm sym}$. In the limit $t \uparrow \infty$, a spin-singlet state with two electrons in the symmetric sector which has finite expectation value of h_0 is written as

$$\Psi = \Big\{\alpha d^\dagger_{0,\uparrow}d^\dagger_{0,\downarrow} + \frac{\beta}{\sqrt{2}}(d^\dagger_{0,\uparrow}d^\dagger_{2,\downarrow} + d^\dagger_{2,\uparrow}d^\dagger_{0,\downarrow}) + \gamma d^\dagger_{2,\uparrow}d^\dagger_{2,\downarrow} + \delta d^\dagger_{1,\uparrow}d^\dagger_{1,\downarrow}\Big\}\tilde\Phi\,, \tag{10}$$

where $\tilde\Phi$ is any state with no electrons in Λ_0. The expectation value of the hopping part of h_0 in this state is given by $\langle h_0^{\rm hop}\rangle = (\Psi, h_0^{\rm hop}\Psi)/(\Psi,\Psi) = A[-2(\lambda^2+2)s|\alpha|^2 - (\lambda^2+2)s|\beta|^2]$, where $A = (|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2)^{-1}$. If we further let $U \uparrow \infty$, a finite energy state must also satisfy $c_{0,\downarrow}c_{0,\uparrow}\Psi = 0$ and $c_{1,\downarrow}c_{1,\uparrow}\Psi = 0$. These conditions lead us to the constraints

$$\frac{\lambda^2}{\lambda^2+2}\alpha + \frac{2\lambda}{(\lambda^2+2)\sqrt{\lambda^2+3}}\beta + \frac{2}{(\lambda^2+2)(\lambda^2+3)}\gamma = 0,$$
$$\frac{1}{\lambda^2+2}\alpha - \frac{\lambda}{(\lambda^2+2)\sqrt{\lambda^2+3}}\beta + \frac{\lambda^2}{2(\lambda^2+2)(\lambda^2+3)}\gamma + \frac{1}{2(\lambda^2+1)}\delta = 0\,. \tag{11}$$

We denote the desired minimum eigenvalue $e_2^{\rm sym}$ for $t = U = \infty$ as $e_{2,\infty}^{\rm sym}$. To get $e_{2,\infty}^{\rm sym}$, we minimize the energy expectation value $\langle h_0^{\rm hop}\rangle$ with respect to the constraints (11). The rest is a tedious but straightforward estimate. Eliminating β from (11), we get $\alpha + \gamma/(\lambda^2+3) + \delta/(\lambda^2+1) = 0$, which implies $|\gamma|^2 + |\delta|^2 \geq f(\lambda)|\alpha|^2$ with $f(\lambda) = \{(\lambda^2+1)^{-2} + (\lambda^2+3)^{-2}\}^{-1}$. By substituting this bound into $\langle h_0\rangle + (\lambda^2+2)s = A(\lambda^2+2)s(-|\alpha|^2 + |\gamma|^2 + |\delta|^2)$, we get $e_{2,\infty}^{\rm sym} + (\lambda^2+2)s \geq A(\lambda^2+2)s|\alpha|^2\{f(\lambda)-1\}$. Noting that $\alpha \neq 0$ in the minimizer, the condition $f(\lambda) > 1$ (which is equivalent to $\lambda > \lambda_c$) implies $e_{2,\infty}^{\rm sym} > -(\lambda^2+2)s$. Since $e_2^{\rm sym}$ is a continuous function of t and U, this proves that $e_2^{\rm sym} > -(\lambda^2+2)s$ for sufficiently large t and U.

By repeating the similar (but easier) variational analysis, we find that $e_2^{\rm as} = -(\lambda^2/3)s > -(\lambda^2+2)s$, and $e_n = \infty$ for $n \geq 3$ when $t = U = \infty$. This implies that the desired condition (9) holds for $\lambda > \lambda_c$ and sufficiently large t and U. The lemma has been proved.

Models in higher dimensions.—Models in higher dimensions can be constructed and analyzed in quite the same spirit [12]. Take, for example, the flatband models studied in [8]. (V and M correspond to $\mathcal{E}$ and $\mathcal{O}$ of the present paper, respectively.) For $x \in V$, we let $a_{x,\sigma}$ as in [8]. For $x = m(v,w) \in M$, we let $a_{x,\sigma} = \lambda c_{x,\sigma} + c_{v,\sigma} + c_{w,\sigma}$. We define the Hamiltonian as

$$H = -s \sum_{\substack{x\in V\\ \sigma=\uparrow,\downarrow}} a_{x,\sigma}^\dagger a_{x,\sigma} + t \sum_{\substack{x\in M\\ \sigma=\uparrow,\downarrow}} a_{x,\sigma}^\dagger a_{x,\sigma} + U \sum_{x\in\Lambda} n_{x,\uparrow} n_{x,\downarrow}, \tag{12}$$

which again contains next-nearest neighbor hoppings and on-site potentials. Equation (12) should be compared with (3). By a straightforward extension of the present method, we can prove that the ground states of the model with $|V|$ electrons exhibit ferromagnetism when λ, t/s, and U/s are sufficiently large [12].

It is a pleasure to thank Tohru Koma, Andreas Mielke, and Bruno Nachtergaele for useful discussions.

*Electronic address: hal.tasaki@gakushuin.ac.jp

[1] D. C. Mattis, *The Theory of Magnetism* (Springer-Verlag, Berlin, 1981).
[2] W. J. Heisenberg, Z. Phys. **49**, 619 (1928).
[3] For a recent survey of rigorous results in the Hubbard model, see E. H. Lieb, in Proceedings of Advances in-Dynamical Systems and Quantum Physics (World Scientific, Singapore, to be published); and in Proceedings of 1993 NATO ASW, The Physics and Mathematical Physics of the Hubbard Model, (Plenum, New York, to be published).
[4] Recently Müller-Hartmann [J. Low. Temp. Phys. **99**, 349 (1995)] argued that the Hubbard model with $U = \infty$ on a one-dimensional zigzag chain exhibits ferromagnetism. Since the model is away from half-filling, $U = \infty$ does not necessarily mean that the model is singular. Although his argument is quite interesting, it does not form a mathematically rigorous proof (as far as we can read from his paper) as it involves an uncontrolled continuum limit of a strongly interacting system. Interestingly, the geometry of the chain is exactly the same as that of ours.
[5] Y. Nagaoka, Phys. Rev. **147**, 392 (1966); D. J. Thouless, Proc. Phys. Soc. London **86**, 893 (1965).
[6] E. H. Lieb, Phys. Rev. Lett. **62**, 1201 (1989).
[7] A. Mielke, J. Phys. A **24**, L73 (1991); **24**, 3311 (1991); **25**, 4335 (1992); Phys. Lett. A **174**, 443 (1993).
[8] H. Tasaki, Phys. Rev. Lett. **69**, 1608 (1992); A. Mielke and H. Tasaki, Commun. Math. Phys. **158**, 341 (1993).
[9] H. Tasaki, Phys. Rev. Lett. **73**, 1158 (1994).
[10] H. Tasaki, J. Stat. Phys. (to be published).
[11] The reader might ask what the basic mechanism of ferromagnetism is in the present models. Our ferromagnetism may be (partially) interpreted in terms either of the standard two pictures of ferromagnetism, namely, the Heisenberg-like "exchange" picture and the band-electron picture. The most crucial point in our proof is that we describe electronic states in such a way that both the particlelike character of electrons and the band structure of the model are taken into account.
[12] H. Tasaki (to be published).
[13] We can of course consider models in which $\mathcal{E}$ sites and $\mathcal{O}$ sites have different values of U.
[14] This special property of the dispersion relations is closely related to the representation (3) of the Hamiltonian.
[15] K. Kusakabe and H. Aoki, Physica (Amsterdam) **194B–196B**, 215 (1994); Phys. Rev. Lett. **72**, 144 (1994).
[16] A support to this conjecture is given by a perturbation theory using the Wannier basis. It suggests that the low energy effective theories of these doped models are ferromagnetic t-J models, where $|t/J|$ can be made arbitrarily small by taking Hubbard models close to flatband models.