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Evidence for Complex Subleading Exponents from the High-Temperature Expansion of Dyson's Hierarchical Ising Model

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Using a renormalization group method, we calculate 800 high-temperature coefficients of the magnetic susceptibility of the hierarchical Ising model. The conventional quantities obtained from differences of ratios of coefficients show unexpected smooth oscillations with a period growing logarithmically and can be fitted assuming corrections to the scaling laws with complex exponents.

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The renormalization group method [1] has enhanced considerably our understanding of elementary processes and critical phenomena. In particular, it has allowed the computation of the critical exponents of lattice models in various dimensions. On the other hand, the critical exponents can be estimated from the analysis of high-temperature series [2]. Showing that the two methods give precisely the same answers is a challenging problem [3]. More generally, much could be gained if we could combine these two approaches, in particular, in the context of lattice gauge theories.

As far as the numerical values of the critical exponents are concerned, there are two difficulties. The first one [4] is that one needs *much* longer high-temperature series than the ones available [5] (which do not go beyond order 25 in most of the cases) in order to make precise estimates. The second is that the practical implementation of the renormalization group usually requires projections into a manageable subset of parameters characterizing the interactions. It is nevertheless possible to design lattice models [6], called hierarchical models, which can be seen as approximate versions of nearest neighbor models, for which such projections are unnecessary. For the hierarchical models, the renormalization group transformation reduces to a recursion formula, which is a simple integral equation involving only the local measure. This simplicity allows one to control rigorously [7] the renormalization group transformation and to obtain accurate estimates of the critical exponents [8]. As explained in the next paragraph, the recursion formula also allows one to calculate the high-temperature expansion to very high order. Consequently, the hierarchical model is well suited to study the questions addressed above.

In this Letter, we report the results of a large scale calculation that performs the high-temperature expansion of the magnetic susceptibility of Dyson's hierarchical Ising model up to order 800. The models considered here have 2^n sites. Labeling the sites with n indices $x_n, \ldots, x_1$, each index being 0 or 1, we can write the Hamiltonian as

$$H = -\frac{1}{2}\sum_{l=1}^{n}\left(\frac{c}{4}\right)^{l}\sum_{x_n,\ldots,x_{l+1}}\left(\sum_{x_l,\ldots,x_1}\sigma_{(x_n,\ldots,x_1)}\right)^2. \qquad (1)$$

The free parameter c that controls the strength of the interactions is set equal to $2^{1-2/D}$ in order to approximate a nearest neighbor model in D dimensions. In the following, the spins $\sigma_{(x_n,\ldots,x_1)}$ are integrated with a local Ising measure; i.e., they only take the values ± 1. The integrations can be performed iteratively using a recursion formula studied in Ref. [7]. Our calculation uses the Fourier transform of this recursion formula with a rescaling of the spin variable appropriate to the study of the high-temperature fixed point [9]. It amounts to the repeated use of the recursion formula

$$R_{l+1}(k) = C_{l+1}\exp\left[-\frac{1}{2}\beta\left(\frac{c}{2}\right)^{l+1}\frac{\partial^2}{\partial k^2}\right]\left[R_l\left(\frac{k}{\sqrt{2}}\right)\right]^2, \qquad (2)$$

which is expanded to the desired order in β. The initial condition for the Ising measure chosen here is $R_0 = \cos(k)$. The constant C_{l+1} is adjusted in such a way that $R_{l+1}(0) = 1$. After repeating n times this procedure, we can extract the finite volume magnetic susceptibility $\chi_n(\beta) = 1 + b_{1,n}\beta + b_{2,n}\beta^2 + \cdots$ from the Taylor expansion of $R_n(k)$ which reads $1 - (1/2)k^2\chi_n + \cdots$. This method has been presented in Ref. [9] and checked using results obtained with conventional graphical methods [10]. In most of the calculations presented below, we have used $n = 100$, which corresponds to a number of sites larger than 10^{30}. The calculations were implemented with a C program, which ran for 6 weeks on a DEC-alpha 3000/4000 in order to obtain 800 coefficients for $D = 3$.

For the discussion that follows, it is crucial to estimate precisely the errors made in the calculation of the coefficients. There are two sources of errors: the numerical roundoffs and the finite number of sites [11]. We claim that with 2^{100} sites and $3 \leq D \leq 4$, the finite volume effects are several orders of magnitude smaller than the roundoff errors. From Eq. (2), one sees that the leading volume dependence will decay as $(c/2)^n$. This observation can be substantiated by using exact results at finite volume [10] for low order coefficients, or by displaying the values of higher order coefficients at successive iterations as in Fig. 1 of Ref. [9]. In both cases, we observed that the $(c/2)^n$ law worked remarkably well. For the main calculation presented below, we have used $c = 2^{1/3}$ (i.e., $D = 3$) and $n = 100$, which gives volume effects of the order of 10^{-20}. On the other hand, the roundoff errors are expected to grow like the square root of the number of arithmetical operations. In Ref. [9], we have estimated that this number was approximately nm^2 for a calculation up to order m in the high-temperature expansion with 2^n sites. Putting this together, we estimated that for $n = 100$, the error on the mth coefficient will be of order $m \times 10^{-16}$. We have verified this approximate law by calculating the coefficients using a rescaled temperature and undoing this rescaling after the calculation. We have chosen the rescaling factor to be 0.8482 and the rescaled critical temperature is then approximately 1. This prevents the appearance of small numbers in the calculation. If all the calculations could be performed exactly, we would obtain the same results as with the original method. However, for calculations with finite precision, the two calculations have independent roundoff errors. Comparing the results obtained with the two methods for the coefficients up to order 200 shows that the numerical fluctuations of b_m grow approximately as $m \times 10^{-16}$. More conservatively, we can say that the numerical errors are bounded by $m \times 10^{-15}$. We conclude that for the calculations reported below, the errors on the coefficients are dominated by the numerical roundoffs, and we estimate that they do not exceed 10^{-12}.

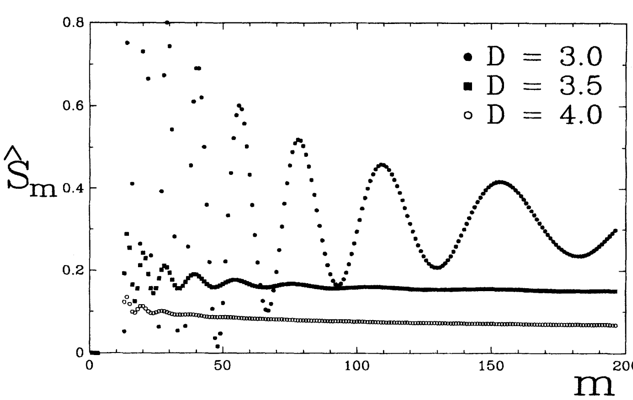

FIG. 1. The extrapolated slope $\hat{S}_m$ for $m \leq 200$ and $D = 3$, 3.5, and 4.

In order to estimate γ, we used standard methods described in Refs. [2,4]. For the sake of definiteness, we recall a few definitions. First, we define $r_m = b_m/b_{m-1}$, the ratio of two successive coefficients. We then define the normalized slope S_m and the extrapolated slope $\hat{S}_m$ as

$$S_m = -m(m-1)(r_m - r_{m-1})/[mr_m - (m-1)r_{m-1}],$$
$$\hat{S}_m = mS_m - (m-1)S_{m-1}. \tag{3}$$

The extrapolated slope, which is free of order n^{-1} corrections [4], is displayed in Fig. 1 for $m \leq 200$. For comparison, we have also displayed the results for $D = 3.5$ and 4. A surprising feature is the clear appearance of large oscillations for $D = 3$. When D is increased, the amplitude of these oscillations diminishes. They are still present at $D = 4$ and can be seen better by plotting $\hat{S}_{m+1} - \hat{S}_m$. One important point of this Letter is to establish that these oscillations are not due to the errors discussed above. As a consequence of the multiplications by m appearing in the definition of the extrapolated slope, the errors are amplified by a factor that can be as large as 10^5 for m near 100 and 10^7 for m near 500. However, even when multiplied by such large factors our most conservative estimate of the numerical errors gives errors on the extrapolated slope that are several orders of magnitude smaller than the amplitude of the oscillations. We have made independent checks of this statement for $D = 3$ by calculating directly the extrapolated slope for $n = 100$ and 200 and by using an intermediate temperature rescaling as explained above. The smoothness of the oscillations appears clearly in Fig. 2, where $\hat{S}_m$ is displayed for $50 \leq m \leq 800$. This smoothness rules out large numerical fluctuations. In conclusion, we have established that the oscillations in the extrapolated slope are a genuine feature of the model considered. Figure 2 also shows that the extrema are not equally spaced. Instead, the location of one extremum can be approximately found by multiplying the location of the previous extremum by 1.19. In other words, the extrema of Fig. 2 would look equally spaced if the abscissa variable had been $\ln(m)$ instead of m. This, of course, suggests the use of a complex exponent since $\mathrm{Re}(m^{i\sigma}) = \cos[\sigma \ln(m)]$.

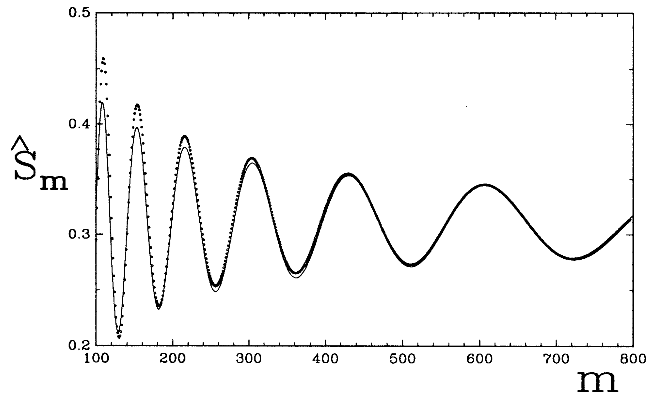

FIG. 2. The dots are the extrapolated slope $\hat{S}_m$ for $50 \leq m \leq 800$ and $D = 3$. The continuous curve is the fit described in the text.

In the conventional description [12] of the renormalization group flow near a fixed point with only one eigenvalue $\lambda_1 > 1$, one expects that the magnetic susceptibility can be expressed as

$$\chi = (\beta_c - \beta)^{-\gamma}[A_0 + A_1(\beta_c - \beta)^{\Delta} + \cdots], \tag{4}$$

with $\Delta = |\ln(\lambda_2)|/\ln(\lambda_1)$ and λ_2 being the largest of the remaining eigenvalues. It is usually assumed that these eigenvalues are real. This implies [4] that

$$\hat{S}_m = \gamma - 1 + Bm^{-\Delta} + O(m^{-2}). \tag{5}$$

If Δ is real, there is no room for the oscillations in this description. Nevertheless, the fact that the period of oscillation increases logarithmically with m suggests that one could modify slightly Eq. (5) by allowing B and Δ to be complex and selecting the real part of the modified expression. This introduces two new parameters, and we have chosen to use the following modified parametrization of the extrapolated slope:

$$\hat{S}_m = \gamma - 1 + Km^{-\rho}\cos\left[2\pi\frac{\ln(m/m_0)}{\ln(\mu)}\right] + O(m^{-2}). \tag{6}$$

This parametric expression allows good quality fits for m large enough. For instance, a least squares fit for the $m \geq 300$ data yields $\mu = 1.412$, $m_0 = 512$, $\rho = 0.67$, $\gamma = 1.310$, and $K = 2.53$. The fit is displayed in Fig. 2. More accurate results could presumably be obtained if we had a consistent description of the oscillations involving definite relations among the parameters of Eq. (6). The value of γ is in good agreement with the result [7,8] obtained with the ϵ expansion, namely, 1.300. The value of ρ is not far from $|\ln(\lambda_2)|/\ln(\lambda_1)$, which is approximately 0.46 according to Refs. [7,8].

We have considered two possible explanations of the oscillatory behavior. Both explanations are compatible with the parametrization of Eq. (6); however, they are completely different from a conceptual point of view. In the first explanation, λ_2 is replaced by a couple of complex conjugated eigenvalues. In the second explanation, the eigenvalues stay real but the constants A_0 and A_1 in Eq. (4) are replaced by periodic functions of $\ln(\beta_c - \beta)$ with period $\ln(\lambda_1)$. We now proceed to discuss each possibility separately.

The replacement of λ_2 by a couple of complex conjugated eigenvalues is a minimal modification that requires no more parameters than the ones introduced in Eq. (6). However, with the exception of a class of triangular Ising models with space dependent couplings [13], complex eigenvalues do not appear in any calculation we know. Let us discuss a few examples where the eigenvalues are real. In the context of field-theoretical calculations, the eigenvalues are extracted from the matrix of derivatives of the beta functions evaluated at a fixed point, and in all the calculations we know real eigenvalues have been obtained. General arguments for this have been found in the case of conformal theories in two dimensions: It has been shown for low order calculations (in the coupling constants) that the matrix mentioned above is symmetric [14], which implies real eigenvalues. Exactly solvable Gaussian models provide further examples of a real spectrum where the largest eigenvalues are widely separated. These properties of the Gaussian spectrum remain valid if one uses the ϵ expansion [1,7,8] with ϵ small enough. This result is of direct relevance to our problem and needs to be discussed in more detail. For the hierarchical Ising model, one could imagine that when D is continuously evolved from 4 (where the fixed point is expected to be Gaussian) to 3, the validity of the ϵ expansion breaks down, and two real eigenvalues merge into each other and subsequently evolve as complex conjugates of each other. However, our numerical data do not support the idea of such a sudden change. As shown in Fig. 1, the amplitude of the oscillations increases with $4 - D$, but the oscillations are already present at $D = 4$. It appears from these considerations that the possibility of complex eigenvalues would require unexpected circumstances such as, for instance, the existence of a new type of fixed point. On the other hand, the fact that our best estimate of γ agrees within 1% with the conventional calculations [7,8] indicates that such a radically different approach might be unnecessary.

The second explanation that we have considered is that the eigenvalues stay real but the constants A_0 and A_1 in Eq. (4) are replaced by functions of $\beta_c - \beta$ invariant under the rescaling of $\beta_c - \beta$ by a factor $(\lambda_1)^l$, where l is any positive or negative integer. This invariance implies that these functions are periodic functions in $\ln(\beta_c - \beta)$ with period $\ln(\lambda_1)$ and can be expanded in integer powers of $(\beta_c - \beta)^{i2\pi/\ln(\lambda_1)}$. This possibility appears naturally for models with one relevant (and real) eigenvalue λ_1, which satisfy a renormalization group equation discussed in Sec. II of Ref. [15]. It is not clear that the susceptibility of the hierarchical model satisfies an equation of this type. However, discrete scale invariance is quite reminiscent of the peculiar symmetries of the hierarchical model discussed in Refs. [10,16]. In the special case where A_0 is restricted to the constant mode and A_1 to a real combination of $(\beta_c - \beta)^{\pm i2\pi/\ln(\lambda_1)}$ we recover a parametrization equivalent to Eq. (6). Our numerical data are in good agreement with the specific prediction concerning the periodicity: The best fit value $\mu = 1.412$ in Eq. (6) is close to the best estimate of the largest (and only relevant) eigenvalue [7,8] $\lambda_1 = 1.427$. In conclusion, the second possibility appears plausible, but it remains to be shown that corrections involving $(\beta_c - \beta)^{i2\pi/\ln(\lambda_1)}$ follow from Eq. (2). This question is presently under study [17].

A complete resolution of the problem is important because one needs to know if the oscillations reported here are due to the peculiarities of the hierarchical model or if these are also present for models with translational invariance. If the oscillations are peculiar to the hierarchical approximation, it is important to determine to which extent they affect results obtained using this approximation. To take an example of great importance in particle physics, an upper bound on the mass of a self-interacting scalar particle, also called triviality bound, was derived [18] using an approximation closely related to the hierarchical approximation considered here. Since the oscillations appear in the susceptibility, which is a physical quantity, they should also be apparent in the flow of bare quantities used to establish triviality bounds. The analysis of Ref. [18] is being reconsidered [17] in this context. On the other hand, if the mechanism responsible for the oscillations applies to models having a conventional translational invariance, it could provide one with a more complete understanding of the "noise" [4], which makes the analysis of high-temperature series difficult. We have analyzed the longest series available for models with translational invariance, namely, the first 54 terms of the high-temperature expansion for the nearest neighbor Ising model on a square lattice [19]. After using a Euler transformation to get rid of the oscillations (of constant period 2) coming from the antiferromagnetic transition at $-\beta_c$, we found damped oscillations in the differences of successive extrapolated slopes. These oscillations decay much faster than in the hierarchical case. The series is unfortunately too short and not regular enough to draw definite conclusions concerning a logarithmic increase of the period of oscillation.

In conclusion, we have shown that a calculational method of the high-temperature expansion based on the renormalization group method can be a very powerful tool when the hierarchical approximation is used. Our analysis of the magnetic susceptibility has shown that unexpected oscillations appear in the extrapolated slope. Our most plausible explanation indicates that these oscillations may be related to the peculiar symmetries of Dyson's hierarchical model. A detailed understanding of these oscillations is required in order to provide a precise comparison between the results obtained from the high-temperature expansion and the ϵ expansion for the hierarchical model, as well as for using and improving the hierarchical approximation for models with a conventional translational invariance.

One of us (Y.M.) stayed at the Aspen Center for Physics during the last stage of this work and benefited from stimulating conversations with the participants, especially with N. Warner.

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