Nonlocal Electron Transport in a Plasma

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We have developed a nonlocal linear theory of electron transport in plasmas with arbitrary electron collisionality. Closure relations for the fluid equations are derived from a solution to the electron Fokker-Planck equation where electron collisions are considered in the limit of large ion charge. We have found nonlocal expressions for electron transport coefficients: the electric conductivity, thermoelectric coefficient, thermal conductivity, electron viscosity, friction, and new transport coefficients related to the ion flow.

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Hydrodynamic equations provide an accurate description for many plasma physics problems. Their form depends upon the transport theory or closure procedure, i.e., the method of expressing higher velocity moments of the distribution function in terms of the hydrodynamical moments which correspond to density, velocity, and temperature of a particular plasma species. The Chapman-Enskog expansion provides such a closure, which is valid for small gradients of hydrodynamical variables in collision dominated plasmas [1]. Unfortunately, this method starts to fail when the inhomogeneity scale length is of the order of hundreds of mean free path lengths. Recent studies have extended the validity of the hydrodynamical equations beyond the Chapman-Enskog approximation into the weakly collisional regime [2–5] or described the collisionless limit [6] by using nonlocal transport coefficients.

In this paper we propose a systematic procedure for a hydrodynamical closure which is valid for arbitrary electron collisionality and for slowly varying processes. Our method includes a solution to a linearized kinetic equation which is based on the two simplifying assumptions of large ionic charge, $Z \gg 1$, and small amplitude perturbations. The method of solution to the kinetic equation combines two approximations from previous studies: Electron-electron (*e-e*) collisions affect only the evolution of the symmetric part of the electron distribution function and are neglected as compared to electron-ion (*e-i*) collisions in the equations for higher order angular harmonics [2–4,7]; the anisotropic part of the electron distribution function includes contributions from all higher order angular harmonics due to the summation method described in Refs. [8,9]. The new transport relations involve nonlocal electrical conductivity, thermal conductivity, thermoelectric coefficient, viscosity, and new transport coefficients in the weakly collisional regime related to the ion flow velocity. In the collisional limit the classical Braginskii's results [1] have been recovered.

The plasma reference state corresponds to a homogeneous Maxwellian distribution function, F_0, with a density n_0 and a temperature T_0. In order to isolate electron transport effects we describe the ions as a cold fluid and neglect ion-ion collisions and energy exchange due to *e-i* collisions. Following previous studies [9,10], we consider the plasma response to a small amplitude periodic perturbation with a wave number $\mathbf{k}$ which may involve a potential ion flow, $\mathbf{u}_i \parallel \mathbf{k}$. The Fourier transformed perturbation (we drop the subscript k for simplicity) of the electron distribution function $f_e(v,\mu,t) = \sum_{l=0}^{\infty} f_l(v) P_l(\mu)$ is expanded in a series of Legendre polynomials $P_l(\mu)$, where μ is the cosine of the angle between $\mathbf{v}$ and $\mathbf{k}$. With this expansion the electron kinetic equation is decomposed into an infinite hierarchy of equations for the harmonics $f_l(v,t)$ of the electron distribution function as derived before in Refs. [8–10]. In the equations for the $l > 0$ harmonics we neglect the time derivatives and *e-e* collision terms as compared to the leading *e-i* collision terms. The time derivative of the symmetric part of the electron distribution function, $f_0(v,t)$, is retained because the Z times more frequent *e-i* collisions affect only the anisotropic part of the distribution function and therefore are not present in this equation for f_0. The higher order ($l > 1$) angular harmonics can be related to f_1 and f_0 using a renormalized *e-i* collision frequency $\nu_1(k,v) = \nu_{ei} H_1(kv/\nu_{ei})$ derived in Refs. [3,8]. The *e-i* collision frequency reads $\nu_{ei}(v) = 4\pi Z n_0 e^4 \Lambda / m_e^2 v^3$, where Λ is the Coulomb logarithm, and the function H_1 is approximated by the expression $H_1(kv/\nu_{ei}) = \sqrt{1 + (\pi kv/6\nu_{ei})^2}$ [8]. This renormalization leads to equations for the first two harmonics of the electron distribution function [8,9]

$$\frac{\partial f_0}{\partial t} + \frac{i}{3} kv f_1 - \frac{i}{3} kv u_i \frac{\partial F_0}{\partial v} = C_{ee}[f_0], \quad (1)$$

$$ikvf_0 + i\frac{e}{m_e} k\phi \frac{\partial F_0}{\partial v} - (\nu_1 - \nu_{ei}) u_i \frac{\partial F_0}{\partial v} = -\nu_1 f_1, \quad (2)$$

which are written in the reference frame moving with

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the ion velocity $\mathbf{u_i}$. The *e-e* collision integral C_{ee} in Eq. (1) has been linearized with respect to the electron Maxwellian distribution function. By solving Eqs. (1) and (2) we can express the electron distribution functions f_0 and f_1 in terms of the electric potential ϕ and ion velocity u_i, which in turn are governed by the Poisson equation and ion fluid equations. In particular, the hydrodynamical moments of the electron distribution functions will also depend on ϕ and u_i. This is contrary to the usual concept of the hydrodynamical description, where density, temperature, ϕ, and u_i are independent variables. This apparent problem does not exist in collision dominated plasmas [1], where in the zero order approximation Eq. (1) reads $C_{ee}[f_0] = 0$ and gives a linearized local Maxwellian distribution function as a solution

$$f_0^{(M)}(v,t) = \left[\frac{\delta n(t)}{n_0} + \frac{\delta T(t)}{T_0}\left(\frac{v^2}{2v_{Te}^2} - \frac{3}{2}\right)\right]F_0(v). \tag{3}$$

Equation (3) introduces δn and δT as coefficients by collision invariants. All other moments can be expressed through these variables using Eq. (2). However, for weakly collisional and collisionless plasmas the left hand side of Eq. (1) could be comparable to the *e-e* collision term and the higher order moments in f_0 can no longer be considered as small.

In order to introduce δn and δT into the solution of Eqs. (1) and (2) we assume that the *initial* perturbation of the electron distribution function has a form analogous to Eq. (3). Thus a solution to the evolutionary equations (1) and (2) will depend on four parameters: ϕ, u_i, $\delta n(0)$, and $\delta T(0)$. The actual hydrodynamical moments $\delta n(t) = 4\pi \int_0^\infty dv\, v^2 f_0$ and $\delta T(t) = (4\pi/3m_e n_0)\int_0^\infty dv\, v^2(v^2 - 3v_{Te}^2)f_0$ can be also written as linear combinations of $\delta n(0)$ and $\delta T(0)$. Using these relations we can eliminate the initial perturbations, $\delta n(0)$ and $\delta T(0)$, from the solution which will then depend only on the hydrodynamical moments of the distribution function at the given time.

Our theory starts with the solution of the initial value problem for Eq. (1). After a Laplace transformation f_1 is obtained from Eq. (2) and substituted into Eq. (1) giving the following equation for the symmetric part of electron distribution function

$$\left(\frac{k^2v^2}{3\nu_1} - p\right)\left(f_0 - \frac{e\phi}{T_0}F_0\right) = -p\frac{e\phi}{T_0}F_0 - iku_i\frac{v^2}{3v_{Te}^2}\frac{\nu_{ei}}{\nu_1}F_0 + C_{ee}[f_0] + f_0(v,0), \tag{4}$$

where p is the Laplace transformation variable. As discussed above, the initial perturbation, $f_0(v,0)$, has the form of Eq. (3). The general solution of the linear inhomogeneous equation (4) can be written as a linear combination of three velocity-dependent base functions

$$f_0 = \frac{e\phi}{T_0}F_0 + \left(\frac{\delta n(0)}{n_0} - p\frac{e\phi}{T_0}\right)\psi^N F_0 + \frac{3}{2}\frac{\delta T(0)}{T_0}\psi^T F_0 - iku_i\psi^R F_0, \tag{5}$$

where the $\psi^\varrho(v)$ satisfy three ($\varrho = N,T,R$) similar equations

$$\left(\frac{k^2v^2}{3\nu_1} + p\right)\psi^\varrho = F_0^{-1}C_{ee}[F_0\psi^\varrho] + S_\varrho. \tag{6}$$

The source terms $S_N = 1$, $S_T = v^2/3v_{Te}^2 - 1$, and $S_R = (v^2/3v_{Te}^2)(\nu_{ei}/\nu_1)$ correspond to the perturbations of density (N), temperature (T), and ion velocity (R). Following the procedure introduced in Ref. [10], Eqs. (6) have been solved numerically using an expansion of the functions ψ^ϱ in Sonine-Laguerre polynomials. This expansion is effective even for weakly collisional plasmas, where it can typically involve 50 polynomials. Numerical solutions have been matched with the asymptotic values of ψ^ϱ in the collisional and collisionless limits [7,9]. The full solution to Eq. (6) can be presented in terms of three moments of the functions ψ^ϱ: $J_\delta^\varrho = (4\pi/n_0)\int_0^\infty v^2 dv\, \psi^\varrho F_0 S_\delta$. It follows from Eq. (6) that $J_\varrho^\delta = J_\delta^\varrho$.

Taking the first two velocity moments of Eq. (5) we find instantaneous perturbations of density and temperature in terms of their initial values. Solving this pair of linear algebraic equations for $\delta n(0)$ and $\delta T(0)$ and substituting the results back into Eq. (5) we obtain the desired expression for the electron distribution function in terms its hydrodynamical moments:

$$f_0 = \frac{e\phi}{T_0}F_0 + \left(\frac{\delta n}{n_0} - \frac{e\phi}{T_0}\right)\frac{J_T^T\psi^N - J_T^N\psi^T}{D_{NT}^{NT}}F_0 + \frac{\delta T}{T_0}\frac{J_N^N\psi^T - J_N^T\psi^N}{D_{NT}^{NT}}F_0 + iku_i\left(\frac{D_{NT}^{RT}}{D_{NT}^{NT}}\psi^N + \frac{D_{NT}^{NR}}{D_{NT}^{NT}}\psi^T - \psi^R\right)F_0, \tag{7}$$

where $D_{\delta\varrho}^{\zeta\varsigma} = J_\delta^\zeta J_\varrho^\varsigma - J_\delta^\varsigma J_\varrho^\zeta$. Using Eq. (7) and expressing the anisotropic part of the electron distribution function in Eq. (2) in terms of the hydrodynamical moments we can construct the closure relations, i.e., we can write expressions for the electron drift velocity $\mathbf{u}_e = \mathbf{u}_i - \mathbf{j}/en_e$, where $\mathbf{j} = -e\int d^3v\,\mathbf{v}f_e$ is the electric current, and the electron heat flux $\mathbf{q}_e = \int d^3v\,\mathbf{v}(m_e v^2/2 - 5/2T_0)f_e$. Following the standard notation of Ref. [1] we write the electron particle and energy fluxes $\mathbf{j}$ and $\mathbf{q}_e$ in terms of generalized hydrodynamical forces: the effective electric field $\mathbf{E}^* = -i\mathbf{k}\phi + (i\mathbf{k}/en_0)(\delta n T_0 + \delta T n_0)$, the temperature gradient $i\mathbf{k}\delta T_e$, and the ion velocity $\mathbf{u}_i$

$$\mathbf{j} = \sigma\mathbf{E}^* + \alpha i\mathbf{k}\delta T_e + \beta_j en\mathbf{u}_i,$$
$$\mathbf{q}_e = -\alpha T_0\mathbf{E}^* - \chi i\mathbf{k}\delta T_e - \beta_q n_0 T_0\mathbf{u}_i, \tag{8}$$

where σ is the electric conductivity, α is the thermoelectric coefficient, χ is the temperature conductivity, and $\beta_{j,q}$ are the ion density flux coefficients. Note that these coefficients are scalar functions of $k\lambda_{ei}$ and p, and therefore our transport relations are nonlocal in time and space:

$$\alpha = -(en_0/m_e k^2 v_{Te}^2)[-p + (J_T^N + J_T^T)/D_{NT}^{NT}],$$
$$\chi = (n_0/k^2)[-5p/2 + (2J_T^N + J_T^T + J_N^N)/D_{NT}^{NT}],$$
$$\sigma = (e^2 n_0/m_e k^2 v_{Te}^2)(-p + J_T^T/D_{NT}^{NT}),$$
$$\beta_q = (D_{NT}^{RT} + D_{NT}^{RN})/D_{NT}^{NT}, \quad \beta_j = 1 - D_{NT}^{RT}/D_{NT}^{NT}. \quad (9)$$

The transport relations (8) possess the Onsager symmetry: The coefficient α is the same in both expressions for $\mathbf{j}$ and $\mathbf{q}_e$. This follows from the symmetry of the coefficients J_Q^S.

For the detailed discussion of the transport coefficients we assume a slow plasma motion, $k^2 v_{Te}^2/\nu_{ei} \gg |p|$, and for this reason temporal nonlocality is neglected by putting $p = 0$ in Eqs. (6) and (9). Transport coefficients found from the numerical solution to Eqs. (6) using the Sonine-Laguerre polynomials expansion [10] are shown in Figs. 1–3. The coefficients σ, α, and χ are normalized to their classical values: $\sigma_0 = 32e^2 n_0 \lambda_{ei}/3\pi m_e v_{Te}$, $\alpha_0 = 16en_0\lambda_{ei}/\pi m_e v_{Te}$, and $\chi_0 = 200 n_0 v_{Te} \lambda ei/3\pi$. They all have similar long wavelength asymptotics: $\sigma = \sigma_0(1 - 19Zk^2\lambda_{ei}^2)$, $\alpha = \alpha_0(1 - 107Zk^2\lambda_{ei}^2)$, and $\chi = \chi_0(1 - 235Zk^2\lambda_{ei}^2)$, which were found from the expansion of Eqs. (6) in the small parameter $Zk^2\lambda_{ei}^2 \ll 1$ (cf. [9]). Note that the thermal conductivity in Fig. 1 deviates from the classical limit for much smaller values of $k\lambda_{ei} \sim 0.06/\sqrt{Z}$ than the other transport coefficients. This is because thermal transport involves higher energy electrons which are less collisional. All transport coefficients demonstrate a significant deviation in the weakly collisional region, $k\lambda_{ei} \sim 1$, from their classical values. They are all inversely proportional to the wave number in the short wavelength limit ($Zk^2\lambda_{ei}^2 \gg 1$)

$$\sigma = \frac{5e^2 n_0}{\sqrt{8\pi}\, m_e k v_{Te}} \frac{1 + 1.8\xi}{1 + 2\xi},$$
$$\alpha = -\frac{en_0}{\sqrt{2\pi}\, m_e k v_{Te}} \frac{1}{1 + 2\xi},$$
$$\chi = \frac{4n_0 v_{Te}}{\sqrt{2\pi}\, k} \frac{1}{1 + 2\xi}, \quad (10)$$

and exhibit a fractional power dependence in the intermediate region, $Zk^2\lambda_{ei}^2 \sim 1$ (cf. [7,9]). The function $\xi(k) = 1.9Z^{2/7}(k\lambda_{ei})^{-3/7}$ has been introduced in Ref. [9] from the asymptotic solution to Eqs. (6) in the short wavelength limit. It accounts for the effect of e-e collisions on slow electrons.

The electric conductivity σ (Fig. 1) follows its asymptotic limit very closely. It is also almost independent of the ion charge Z. The thermal conductivity χ is the most sensitive function of Z and the fractional power asymptotics $\propto (k\lambda_{ei})^{-4/7}$ is also visible in Fig. 1. The thermoelectric coefficient changes sign in the intermediate region of $k\lambda_{ei} \sim 1-5$ (cf. inset in Fig. 2).

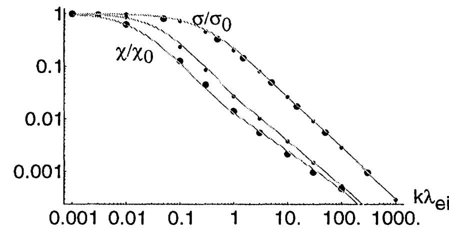

FIG. 1. The wavelength dependence of the electric conductivity σ and the temperature conductivity χ for a plasma with $Z = 8$ (small dots) and $Z = 64$ (big dots). Light lines are analytical approximations.

The closure relations (9) involve new terms which are dependent on the ion velocity $\mathbf{u}_i$ and are related to higher order angular harmonics of the electron distribution function. The coefficients β shown in Fig. 3 contribute to the momentum relaxation due to e-i collisions. Their long wavelength asymptotics, $k\lambda_{ei} \ll 1$, are $\beta_j = 33k^2\lambda_{ei}^2$ and $\beta_q = 133k^2\lambda_{ei}^2$. In the short wavelength limit $\beta_q \propto \ln(k\lambda_{ei})/k\lambda_{ei}$ and disappears and $\beta_j = 1 - O[\ln(k\lambda_{ei})/k\lambda_{ei}]$ approaches unity. Physically this corresponds to the separation between electron and ion dynamics in the collisionless limit, where there are no contributions from ions to the electric current and electron heat flux. We will demonstrate below the importance of coefficients β for the proper description of electron heat flux inhibition and electron Landau damping of ion acoustic waves.

Transport theory is often applied to the case of zero current, which describes quasineutral plasma motions. The generalized Ohm's law in Eq. (8) can be used to eliminate the ambipolar electric field from the following expression for the electron heat flux

$$\mathbf{q}_e = -\kappa i\mathbf{k}\delta T_e - \beta n_0 T_0 \mathbf{u}_i, \quad \kappa = \chi - \alpha^2 T_0/\sigma,$$
$$\beta = \beta_q - e\alpha\beta_j/\sigma, \quad (11)$$

where the two contributions to the electron heat flux are related to the temperature gradient and ion velocity. The ion velocity contribution has not been accounted

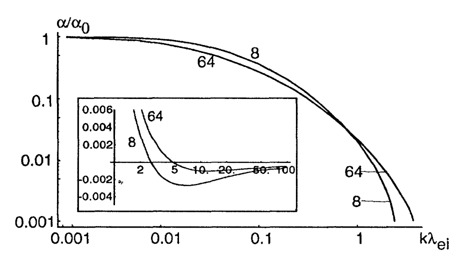

FIG. 2. The wavelength dependence of the thermoelectric coefficient α for plasma with $Z = 8$ and $Z = 64$. The short wavelength region where $\alpha(k)$ changes sign is shown in the inset.

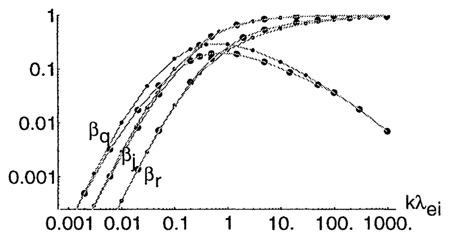

FIG. 3. The wavelength dependence of the ion flux transport coefficients β for a plasma with $Z = 8$ (small dots) and $Z = 64$ (big dots). Light lines are analytical approximations.

for explicitly in previous studies of nonlocal electron transport. For example, the definition of a nonlocal electron heat conductivity in Refs. [8,9] has been derived from ion acoustic perturbations and was defined as the ratio between the electron heat flux and the temperature gradient. The ambiguity in this definition has already been mentioned in Ref. [10]. From Eq. (11) one can see that the definition of the heat conductivity used in [8,9] corresponds to $\kappa + \beta n_0 T_0 u_i / ik\delta T_e$. The second term is negligible in the collisional limit, but in weakly collisional and collisionless regions both terms are comparable.

In addition to the quantities discussed above, hydrodynamics should also include the electron stress tensor $\Pi_{lj} = \int d^3v\, m_e(v_l v_j - \delta_{lj}v^2/3)f_e$ and the friction force between ions and electrons $\mathbf{R}_{ie} = \int d^3v\, m_e \mathbf{v} \nu_{ei} f_e$. The electron stress tensor is related to the friction force $\Pi_{lj} = (3i/4k^2)(k_l\hat{R}_j + k_j\hat{R}_l - \frac{2}{3}\delta_{lj}\mathbf{k}\hat{\mathbf{R}})$, where $\hat{\mathbf{R}} = \mathbf{R}_{ie} - en_0\mathbf{E}^* - 2n_0 i\mathbf{k}\delta T_e$. The friction force can be written as a function of the generalized forces and ion velocity

$$\mathbf{R}_{ie} = -(1 - \beta_j)n_0 e\mathbf{E}^* + \beta_q n_0 i\mathbf{k}\delta T_e - \beta_r m_e n_0 \mathbf{u}_i v_{Te}/\lambda_{ei}, \quad (12)$$

$$\beta_r = 1 + k^2 v_{Te}\lambda_{ei} \times [J_R^R - J_R^N(1 - \beta_j) - J_R^T(1 - \beta_j - \beta_q)] - (2\pi)^{3/2}\frac{v_{Te}}{n_0}\int_0^\infty \frac{dv\, vF_0(v)}{H_1(kv/\nu_{ei})}. \quad (13)$$

The k dependence of β_r (cf. Fig. 3) is similar to β_j and has similar asymptotics. In the collisional limit it disappears as $k^2\lambda_{ei}^2$ and the friction force $\mathbf{R}_{ie} \approx -n_0 e\mathbf{E}^*$ agrees with the classical hydrodynamical theory [1]. In the collisionless limit β_r approaches unity and the friction force becomes proportional to the ion velocity.

The closure relations (8) and (12) combined with fluid equations for electrons and ions provide a description of plasma processes equivalent to the kinetic equations for arbitrary collisionality. As an example, we have derived the damping rate of ion acoustic waves for $k\lambda_{ei} \gg c_s/v_{Te}$:

$$\frac{\gamma_s}{kc_s} = \frac{n_0 c_s}{2k}\left[\frac{(1-\beta)^2}{\kappa} + \frac{e^2\beta_j^2}{T_0\sigma} + \frac{\beta_r}{n_0 v_{Te}\lambda_{ei}}\right], \quad (14)$$

where c_s is the ion acoustic velocity. Note that all nonlocal transport coefficients contribute to ion acoustic damping. Expression (14) agrees very well with the numerical solution to the Fokker-Planck kinetic equation [8] and the analytical theory of Refs. [9,10]. The damping coefficient has the proper hydrodynamic form in the long wavelength limit $k\lambda_{ei} \ll 1$ and takes the form of a collisionless Landau damping $\gamma/kc_s = \sqrt{\pi/8}c_s/v_{Te}$ in the short wavelength region $k\lambda_{ei} \gg 1$.

In summary, we have developed electron nonlocal closure relations for fully ionized plasmas which are valid for arbitrary collisionality. Our theory is linear and restricted to small amplitude perturbations. The wavelength dependence of the transport coefficients is presented here in graphical form, but simple polynomial fits could also be found. New transport coefficients have been introduced, describing the effect of ion flow on particle and energy fluxes.

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