Equation of State for the SU(3) Gauge Theory

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By investigating the SU(3) gauge theory thermodynamics on lattices of various sizes we can control finite lattice cutoff effects. We calculate the pressure and energy density on lattices with temporal extent $N_\tau = 4$, 6, and 8 and spatial extent $N_\sigma = 16$ and 32, and extrapolate to the continuum limit. We find a deviation from ideal gas behavior of (15–20)%, even at temperatures as high as $T \sim 3T_c$. A calculation of the critical temperature for $N_\tau = 8$ and 12 and the string tension for $N_\tau = 32$ is performed to fix the temperature scale, yielding $T_c/\sqrt{\sigma} = 0.629(3)$ in the continuum limit.

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Reaching a quantitative understanding of the equation of state (EOS) of QCD is one of the central goals in finite temperature field theory. The intuitive picture of the high temperature phase of QCD behaving like a gas of weakly interacting quarks and gluons is based on leading order perturbation theory. However, the well-known infrared problems of QCD lead to a poor convergence of the perturbative expansion of the thermodynamic potential even at temperatures very much higher than T_c [1].

At finite temperature, the Euclidean time extent of the system is fixed by the temperature T. Correspondingly, lattice calculations are performed on asymmetric lattices of size $N_\sigma^3 \times N_\tau$ with $N_\sigma \gg N_\tau = 1/aT$, where a is the lattice spacing. So far calculations of bulk thermodynamic quantities have, in general, been carried out on lattices with $N_\tau = 4$ [2]. A small N_τ is a severe limitation, which leads to quite large cutoff effects in thermodynamic quantities. In the thermodynamic limit, $N_\sigma \rightarrow \infty$, these corrections are of $O(N_\tau^{-2})$. The energy density (ϵ) and pressure (p) of an ideal gluon gas are then given by [3]

$$\frac{\epsilon}{T^4} = \frac{3p}{T^4} = (N^2 - 1)\left[\frac{\pi^2}{15} + \frac{2\pi^4}{63}\frac{1}{N_\tau^2} + O\left(\frac{1}{N_\tau^4}\right)\right]. \quad (1)$$

The cutoff effects result from the discretization of the field strength tensor, which introduces $O(a^2)$ deviations from its continuum counterpart. In the case of a free gas it is found that the corrections are as large as 50% for $N_\tau = 4$.

In order to compare lattice calculations of the EOS with continuum perturbation theory or phenomenological models one has to control these finite cutoff effects. This requires a systematic analysis of thermodynamic quantities on lattices with varying N_τ, which then allows an extrapolation of the numerical results to the continuum limit ($N_\tau \rightarrow \infty$). We have carried out such an investigation in pure SU(3) gauge theory, i.e., in the quenched approximation of QCD. For this purpose, one needs high precision results for the action density on asymmetric finite temperature lattices and, in addition, on symmetric, zero temperature lattices of size N_σ^4. All basic thermodynamic quantities can then be calculated from the difference of action densities at zero (S_0) and finite (S_T) temperature [4], which are proportional to the plaquette expectation values, $S_{0(T)} = 6\langle 1 - \frac{1}{3}\mathrm{Tr}U_1U_2U_3U_4\rangle$. We define ΔS as N_τ^4 times this difference, $\Delta S = N_\tau^4(S_0 - S_T)$. One also needs to know the dependence of the physical temperature on the bare gauge coupling, $T^{-1} = N_\tau a(g^2)$.

For our simulations we use an over-relaxed heat-bath algorithm with 5–9 over-relaxation updates followed by one heat-bath update ($\equiv$ one iteration). At several values of the gauge coupling we have performed between 20 000 and 30 000 iterations on lattices of size $16^3 \times 4$ and $32^3 \times N_\tau$ with $N_\tau = 6$ and 8. Note that this implies a ratio $N_\sigma/N_\tau \geq 4$, which is sufficiently large to have reached the thermodynamic limit in the plasma phase [3]. The zero temperature simulations were done on 16^4 and 32^4 lattices with typically 5000 to 10 000 iterations. The temperature scale is determined through calculations of the critical couplings of the deconfinement transition on lattices with $N_\tau = 4$, 6, 8, and 12 and a calculation of the string tension on 32^4 lattices at these critical couplings.

The temperature scale.—Asymptotically, for large values of $\beta = 6/g^2$, the temperature $T = 1/N_\tau a(\beta)$ is given by the perturbative scaling relation $a(\beta)\Lambda_L = R(\beta)$, with

$$R(\beta) = \left(\frac{8\pi^2\beta}{33}\right)^{51/121} \exp[-4\pi^2\beta/33]. \quad (2)$$

At lower β values, this relation between the lattice cutoff a and the coupling g^2 receives corrections and needs to be obtained through the calculation of a physical quantity in units of a, e.g., the string tension or the critical temperature. Different observables may then lead to relations $a(g^2)$, which differ from each other by $O(a^2)$ terms. However, it seems that these differences are small for intermediate values of the gauge coupling. This is seen in the small g^2 dependence of ratios of physical observables. In any case, if one chooses a particular relation $a(g^2)$, obtained from one physical observable, all $O(a^2)$ corrections will drop out in the continuum limit.

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TABLE I. String tensions calculated at the critical couplings for the deconfinement transition, $\beta_c(N_\tau)$. For $N_\tau = 4$ and 6 we evaluate σa^2 at the infinite volume critical coupling using an interpolation of values from Ref. [8]. For $N_\tau = 8$ and 12 we have calculated the string tension at the finite volume critical couplings. The systematic error is also given in these cases.

N_τ	β_c	$\sqrt{\sigma}\, a$	$T_c/\sqrt{\sigma}$
4	5.6925 (2)	0.4179 (24)	0.5983 (30)
6	5.8941 (5)	0.2734 (37)	0.6096 (71)
8	6.0609 (9)	0.1958 (17)	0.6383 (55) (+13)
12	6.3331 (13)	0.1347 (6)	0.6187 (28) (+42)

Here we will fix the relation between a and g^2 through a calculation of the critical temperature. The critical couplings have been extracted from the locations of peaks in the Polyakov loop susceptibility [5]. For the $N_\tau = 4$ and 6 lattices our analysis of the critical couplings is in complete agreement with earlier high statistics calculations [6]. For $N_\tau = 8$ and 12 we find significantly larger values than those obtained in previous calculations [7] on smaller spatial lattices. A comparison shows, however, that our result is consistent with the expected shift towards larger values due to the larger spatial volume used here.

The absolute temperature scale will be fixed through a determination of the string tension on 16^4 and 32^4 lattices at the critical couplings $\beta_c(N_\tau)$. We have obtained the string tension from an analysis of heavy quark potentials calculated from smeared Wilson loops [5]. For $N_\tau = 4$ and 6 the ratio $T_c/\sqrt{\sigma}$ has been determined at the critical couplings extrapolated to the infinite volume limit. For $N_\tau = 8$ and 12 we evaluate this ratio at the critical couplings obtained on lattices with finite N_σ/N_τ. From the volume dependence of the critical couplings studied in Ref. [6] we expect that the infinite volume critical couplings will be larger by about 0.0017 for $N_\tau = 8$ and 0.0057 for $N_\tau = 12$. We therefore systematically underestimate the ratio $T_c/\sqrt{\sigma}$ in these cases. The expected systematic error due to this effect has been estimated by assuming an exponential scaling of $\sqrt{\sigma}\, a$ according to the asymptotic renormalization group equation.

The results for $T_c/\sqrt{\sigma}$ are summarized in Table. I. We have extrapolated the results for the different N_τ values to the continuum limit using a fit of the form $a_0 + a_2/N_\tau^2$. This yields

$$\frac{T_c}{\sqrt{\sigma}} = 0.625 \pm 0.003(+0.004)\,. \qquad (3)$$

The number in parentheses indicates the systematic shift we expect from the infinite volume extrapolation of the critical couplings. We note that the ratio $T_c/\sqrt{\sigma}$ does not show any significant finite cutoff corrections and the parameter a_2 in our fit is compatible with zero within errors. Our estimate of $T_c/\sqrt{\sigma}$ is about 10% larger than earlier estimates [9], which is due to our newly determined critical couplings for the larger lattices. Using $\sqrt{\sigma} = 420$ MeV we find a critical temperature of about 260 MeV.

The lattice cutoff, extracted from the location of the critical couplings, shows the well-known deviations from the asymptotic scaling relation, Eq. (2). This is consistent with recent high statistics results for the $\Delta\beta$ function [10], which describes the change in β needed to change the cutoff by a factor of 2. To parametrize the relation between cutoff and gauge coupling such that these Monte Carlo results as well as the critical temperatures are reproduced we have tried several alternatives. We started with a parametrization of the form $a\Lambda_L = R(\beta_{\rm eff}) \times 0.4818$ [9] where $\beta_{\rm eff}$ is a renormalized coupling [11] defined through the action expectation value $\beta_{\rm eff} = 3(N^2-1)/2S_0$. Since T_c and the $\Delta\beta$ values below $\beta \lesssim 7$ were not too well reproduced by this, we used instead the ansatz $a\Lambda_L = R(\beta)\lambda(\beta)$, where $\lambda(\beta)$ was interpolated as in Ref. [3], and, in addition, adjusted to coincide with the effective coupling parametrization for $\beta \gtrsim 7$ [5].

Equation of state.—Our calculation of thermodynamic quantities is based on a direct evaluation of the free energy density in large spatial volumes, i.e., close to the thermodynamic limit. This requires a numerical integration of the difference of action densities,

$$\frac{p}{T^4}\Big|_{\beta_0}^{\beta} = -\frac{f}{T^4}\Big|_{\beta_0}^{\beta} = N_\tau^4 \int_{\beta_0}^{\beta} d\beta' (S_0 - S_T)\,. \qquad (4)$$

The above relation gives the pressure (free energy density) difference between two temperatures corresponding to the two couplings β_0 and β.

Making use of basic thermodynamic relations we can then evaluate the energy density in the thermodynamic limit from

$$\frac{\epsilon - 3p}{T^4} = T\frac{\partial}{\partial T}(p/T^4) = -6N_\tau^4 a \frac{\partial g^{-2}}{\partial a}(S_0 - S_T)\,, \qquad (5)$$

where the derivative $a\partial g^{-2}/\partial a$ is obtained from our explicit parametrization of $a(g^2)$.

In Fig. 1 we show ΔS for $N_\tau = 8$, which is statistically the most difficult case. For a calculation of the pressure we have to integrate the action densities with respect to β, Eq. (4). For this purpose we use interpolations as shown in Fig. 1. As can be seen from the figure, ΔS rapidly becomes small below the critical coupling. We thus can use a value β_0 close to the critical coupling to normalize the free energy density. Results obtained for the pressure are shown in Fig. 2(a). The observed cutoff dependence reflects the N_τ dependence of the free gluon gas. Quantitatively, however, we find that the cutoff dependence of the pressure is considerably weaker than suggested by the ideal gas calculation.

Errors for the pressure as a function of temperature arise from ambiguities in determining the temperature

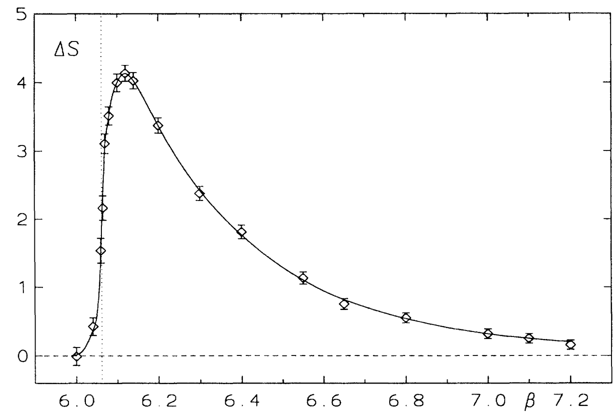

FIG. 1. Difference of action densities ΔS for $N_\tau = 8$ and spatial lattice size $N_\sigma = 32$. The vertical line shows the location of the critical coupling.

scale as well as from our interpolating curves for the action densities. To estimate the interpolation errors, we have integrated ΔS also by using the trapezoidal rule. The resulting differences are on the level of a few percent. They are shown as typical error bars in Fig. 2(a). The ambiguities arising from the choice of parametrization of the temperature scale only amount to a shift in T. This effect is largest for $N_\tau = 4$ and is shown as a dashed curve in Fig. 2(a), for which the temperature scale from the effective coupling scheme was used.

A similar analysis was carried out for $(\epsilon - 3p)/T^4$. Results are shown in Fig. 2(b). Also here we have examined the systematic errors arising from the different parametrizations of $a(g^2)$. For $N_\tau = 4$ these errors are about 6% on the peak of $(\epsilon - 3p)/T^4$ and less than 2% everywhere else. For $N_\tau = 6$, 8 the errors are on the 2% level. The energy density is then obtained by combining the results for p and $\epsilon - 3p$.

Based on the analysis of pressure and energy density at various N_τ values we attempt to extrapolate these quantities to the continuum limit. In the case of a free theory the leading N_τ^{-2} corrections to the continuum limit provide a good description of the actual N_τ dependence only for $N_\tau \geq 6$. This is seen qualitatively also in our numerical data. To extrapolate to the continuum limit, we therefore use the $N_\tau = 6$ and 8 data only, following Eq. (1),

$$\left(\frac{p}{T^4}\right)_a = \left(\frac{p}{T^4}\right)_0 + \frac{c_2}{N_\tau^2}\,. \qquad (6)$$

The systematic error in $(p/T^4)_0$ was estimated by comparing different parametrizations of the temperature scale. A similar analysis has been performed for the energy density.

The extrapolations of pressure, energy, and entropy density are presented in Fig. 3. Note that the curves shown are continuous at T_c, although we have clear indications for a first order transition also from our calculations for $N_\tau = 8$. Since on a finite spatial lattice there are no discontinuities we have averaged over the metastabilities. However, we show the expected size of the discontinuity in the energy density in the thermodynamic limit [6] as a hatched vertical band in Fig. 3. Over a wide temperature range we find that the difference between the extrapolated values and the results for $N_\tau = 8$ is less than 4%, whereas the corresponding difference for the ideal gas is about 8%. Consequently, at fixed N_τ, the ratio between the numerical data and the ideal gas limit for, e.g., the energy density, is not a good approximation to the continuum limit. For instance, at $T = 4T_c$, the ratio is changing from 0.72 at $N_\tau = 4$ to 0.86 at $N_\tau = 8$, whereas the extrapolated result is 0.89. Thus, the extrapolated results are closer to the ideal gas limit than expected on the basis of the $N_\tau = 4$ results.

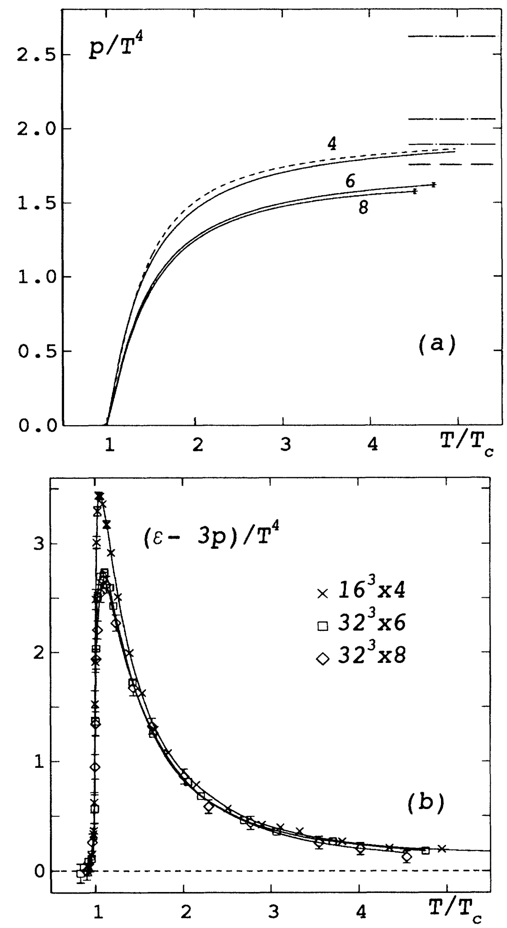

FIG. 2. (a) The pressure versus T/T_c for $N_\tau = 4$, 6, and 8, integrating the interpolations for the action density. Solid curves show our parametrization. The dashed curve for $N_\tau = 4$ is the result of using the temperature scale from the effective coupling scheme. The horizontal dash-dotted lines show the ideal gas values for $N_\tau = 4$, 6, and 8; the horizontal dashed line is the continuum value. In (b) we show the difference $(\epsilon - 3p)/T^4$.

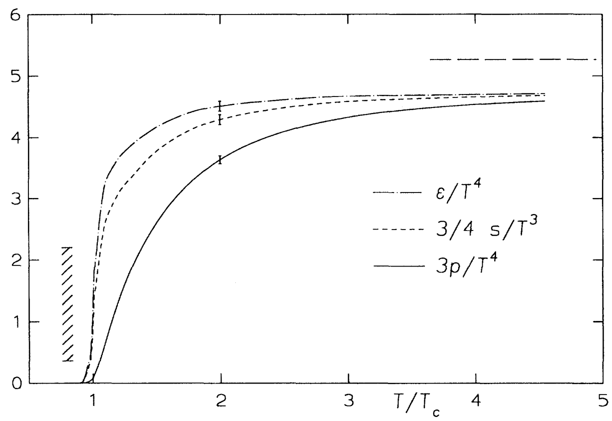

FIG. 3. Extrapolation to the continuum limit for the energy density, entropy density, and pressure versus T/T_c. The dashed horizontal line shows the ideal gas limit. The hatched vertical band indicates the size of the discontinuity in ϵ/T^4 (latent heat) at T_c [6]. Typical error bars are shown for all curves.

There are two general features of the extrapolated results for the equation of state of a gluon gas. First, the energy density rises rapidly to about 85% of the ideal gas value at $2T_c$ and then shows a rather slow increase, which is consistent with a logarithmic behavior as one would expect from a leading order perturbative correction. Second, the pressure rises much more slowly near T_c and even at $T \simeq 3T_c$ shows sizable deviations from the ideal gas relation $\epsilon = 3p$.

The trace anomaly is related to the difference between the gluon condensate at zero and finite temperature [12], $\epsilon - 3p = G^2(0) - G^2(T)$. At the peak position of $(\epsilon - 3p)/T^4$ at $T \simeq 1.1T_c$, one has $(\epsilon - 3p)_{\text{peak}} = (0.57 \pm 0.02)\sigma^2 \simeq 2.3\ \text{GeV/fm}^3$, which should be compared with the value of the zero temperature gluon condensate, $G^2(0) \simeq 2\ \text{GeV/fm}^3$ [12]. This fulfills the above relation if $G^2(T) \simeq 0$ at $T \simeq 1.1T_c$.

To conclude, we have carried out a systematic investigation of thermodynamic quantities on different size lattices. This allowed us to analyze finite cutoff effects. For the first time, from lattice calculations of the pure SU(3) gauge theory, we thus obtained results for thermodynamic quantities in the continuum limit.

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