Controlling Complexity

Leon Poon and Celso Grebogi
Institute for Plasma Research, University of Maryland, College Park, Maryland 20742
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Complex systems have the property that many competing behaviors are possible, and the system tends to alternate among them. In fact, the ability of a complex system to access many different states, combined with its sensitivity, offers great flexibility in manipulating the system's dynamics to select a desired behavior. By understanding dynamically how some of the complex features arise, we show that it is indeed possible to control a complex system's behavior. This is illustrated by using the noisy double rotor map as a paradigm.

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Scientists have come to the realization that many naturally occurring systems are neither completely ordered and predictable nor completely random and unpredictable. In fact, the behavior of many systems in nature fall between these two opposite ends. The term "complexity" has been coined to denote the study of "complex systems," that is, systems with complicated and intricate features having both elements of order and elements of randomness. They arise in fields as diverse as biology, chemistry, computer science, geology, physics, and fluid mechanics. Some systems that exhibit apparent complex behavior are Rayleigh-Bernard convection [1], an extended optical system [2], neuronal activity [3], and fluidized beds [4].

But what is a complex system? While there is no general consensus on what constitutes a good quantitative measure of complexity, the following general traits are some that most agree a complex system often exhibits [5]. (i) A complex system is composed of many parts that are interrelated in a complicated manner. Usually, these intricate mutual relations result in some form of coherent structure. (ii) A complex system possesses both "ordered" and "random" behaviors. (iii) A complex system often exhibits a hierarchy of structures; that is, nontrivial structures exist over a wide range of time and/or length scales.

These complex features are generally, but not necessarily, found in systems with many degrees of freedom. Instead of evolving towards one dominating attracting set, which is common in lower dimensional systems, the interactions among the large number of attracting and unstable sets often result in rich and varied dynamics where many competing behaviors are possible. As a result, the dynamics of complex systems tend to alternate among these different behaviors and which of them are observed at a given time are often sensitive to minor perturbations. These two key attributes, accessibility to many states and sensitivity, present us with an opportunity to influence and manipulate a complex system's dynamics.

In this paper, we show how complexity can be realized and how its dynamics can be manipulated using small perturbations. Specifically, we use the double rotor system as a paradigm of a relatively low-dimensional dynamical system (as opposed to an extended system), which exhibits many characteristics typical of complex systems when it is subjected to random external noise. We argue that the three traits mentioned above are apparent in many of its behaviors. Furthermore, we also show how to use small amplitude feedback control to influence and manipulate the behavior of the system such that its trajectories, which formerly traversed the various attracting and unstable chaotic sets, will now be confined to a neighborhood of one of the attracting states of our choice.

The double rotor map is a rich dynamical system with many complex features [6]. For a wide range of parameters, the double rotor map has a multitude of periodic attractors (the coexistence of tiny chaotic attractors is also possible for some parameters). Moreover, only a small portion of the basins of attraction for these periodic attractors have "smooth" structure; that is, only initial conditions within small balls centered at each component of the periodic attractor unambiguously asymptote to their respective attractors. With the exception of these small open neighborhoods about the periodic attractors, the majority of phase space is occupied by fractal basin boundaries whose dimension ($\approx$3.998 for the parameters we study) is very close to the dimension of the phase space.

The complicated basin structures of the double rotor system plays an important role in the system's complex dynamics. The presence of unstable invariant sets embedded in the fractal basin boundaries has two appreciable effects on the system: long chaotic transient behavior and final state sensitivity [7]. Typical trajectories with initial conditions near the boundaries (which is highly probable since most of the phase space is dominated by basin boundaries) will undergo chaotic motion for long times before settling on one of the periodic attractors. The dynamics is then characterized by a large number of periodic attractors "embedded" in a sea of transient chaos. The fine scale intermingling among the various basins makes it very difficult to predict the future state of trajectories for arbitrary initial conditions [7]. Thus, the system is extremely sensitive to perturbations.

Because fractal basin boundaries permeate most of the phase space, the addition of small amplitude noise prevents the trajectories from settling into any of the stable

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periodic behavior. What happens instead is that a trajectory will come close to one of the periodic attractors and stay in its neighborhood for some time. For this period of time, the trajectory's behavior is governed by the periodic attractor, and it is, thereby, ordered. If one takes a large number of trajectories near this periodic attractor and then a snapshot is taken after the noisy system has evolved for some time, one would see coherent structures in phase space in the vicinity of the periodic attractor as shown in Fig. 1. (Coherent structures are colored with darker shades. What the picture represents and how it is generated will be discussed shortly.) However, this ordered behavior, for a particular trajectory, is transitory, and noise will eventually move the trajectory out of this state into the fractal boundary region. The trajectory will then spend some amount of time within the massive basin boundary region executing an apparently chaotic motion before approaching the same or another periodic attractor. The period of time in the fractal basin boundaries corresponds to the trajectory's "random" behavior. For an ensemble of trajectories, these random structures are represented by the lighter shades in Fig. 1. Hence, a typical noisy trajectory alternates between intervals of chaotic motion and intervals of nearly periodic behavior as shown in Fig. 2. A physical system that suggests the above behavior was observed by Bergé and Dubois in a Rayleigh-Bernard convection experiment [1].

The dynamics we have just described correspond to traits (i) and (ii) expected in complex systems. The fractal basin boundaries with the embedded chaotic sets provide the link among the various attractors. The presence of both "ordered" and "random" behaviors mentioned in (ii) is evidenced by the trajectories of our system cascading from the random structures down to the coherent structures. Noise, on the other hand, displaces trajectories from the coherent structures back to the random structures. Therefore, dynamically there is such an interplay between random and coherent structures that the system is neither completely predictable nor completely random.

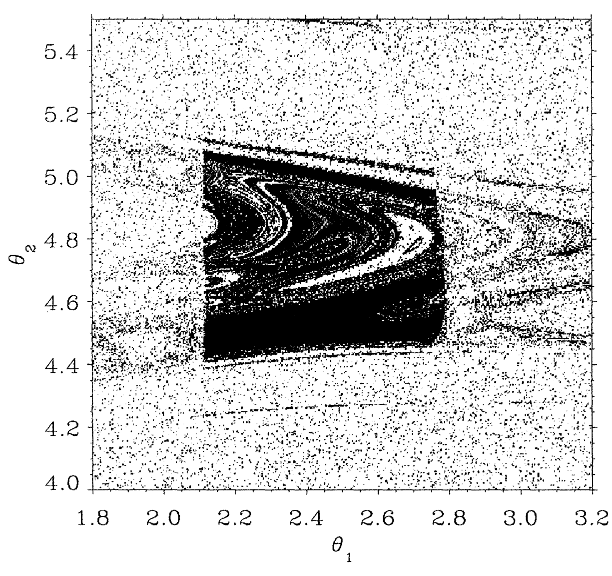

FIG. 1(color). A blowup of a region that exhibits a combination of coherent and random structures. The picture is generated after 100 iterations of the noisy map. The apparent sharp vertical boundaries in the picture is an artifact of projecting a four-dimensional figure onto a two-dimensional space, namely, $(\theta_1, \theta_2, 2.24, -3.65)$.

The coherent structures mentioned above and shown in Fig. 1 are found by examining the dynamics of a group of trajectories (10^5) near one of the periodic orbits. We follow the evolution of an ensemble of initial conditions in physical space (varying only θ_1 and θ_2) near one of the periodic attractors while the system is subjected to random noise with uniform distribution. In order to get an indication of whether a trajectory is evolving chaotically or has settled into some sort of periodic behavior after a certain number of iterates, say n, we compute the largest eigenvalue of the Jacobian matrix for each of the trajectories in the ensemble at the next iterate, i.e., $n + 1$. The largest eigenvalue is a measure of the local instability, and hence it indicates the "jump" the trajectory will make from its current location. We then assign a color to a point corresponding to the trajectory's initial condition according to how large a trajectory "jumped" at the $(n + 1)$th iterate, where the lighter the shade the bigger the jumps. Thus, a trajectory point with the largest eigenvalue less than 1 would be various shades of brown, while a trajectory point with the largest eigenvalue more than 1 would be various shades of white. In Fig. 1, we see an intricate picture exhibiting coherence and randomness. Thus, the combination of deterministic chaos and stochastic noise has resulted in the emergence of coherent structures.

We can quantify order and randomness by encoding the dynamics of the system into symbolic sequences. We choose to encode the trajectory by the sequence in which it visits the neighborhood of the attractors. Of course, the trajectory visits the different attracting sets by traversing through the chaotic sets in the boundaries. These chaotic excursions are implicit in our choice of the alphabet, each

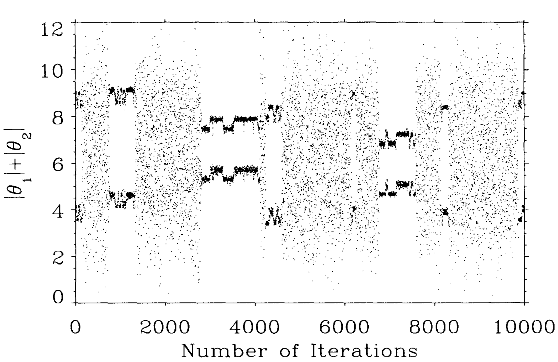

FIG. 2. Time series of a typical noisy trajectory.

symbol corresponding to an attractor. For the parameters we study, eight symbols are needed in the alphabet [8]. We compute now the "transition" probabilities among the various attracting sets. The complex dynamics can now be characterized by the Kolmogorov-Sinai (KS) entropy [9]

$$h = \lim_{n\to\infty} \frac{H_n}{n} = \lim_{n\to\infty} \frac{1}{n}\left(-\sum_{|S|=n} p(S)\ln p(S)\right),$$

where $S = s_1 s_2 \cdots s_n$ denotes a finite symbol sequence; $p(S)$, the relative frequency of S; and H_n, block entropy of block length n. Since $s_i \in \{1, \ldots, 8\}$, sequences where the letters are completely uncorrelated would have a KS entropy value of $\ln 8 \simeq 2.08$, while a periodic sequence would yield a value of 0. Our computation shows that H_n/n converges fairly rapidly to a value of $h \approx 1.42$. KS entropy can be thought of as a measure of the coherence of a trajectory as it evolves, so an intermediate value of h means not only that the symbolic sequence is unpredictable (since $h > 0$), but there is also structure in the set of all possible symbol sequences. It should be noted that the encoding scheme we have implemented does not take into account the amount of time the trajectory spends in the ordered as well as the chaotic regions.

A consequence of the complex interplay between the coherent and random structures, and the irregular switchings among them, is the appearance of nontrivial length and/or time scalings in the noisy double rotor system, a quality [trait (iii)] that we expect in complex systems. First, there is the length of the chaotic transients that is a measure of how long a trajectory spends in the vicinity of each chaotic saddle embedded in the fractal basin boundary. It is known that the average length of a chaotic transient is related to the dimension and the Lyapunov exponents of the chaotic saddle [10]. Each chaotic saddle, in general, contributes a distinct time scale, and the overall chaotic transient $\langle\tau\rangle$ would then be a conglomeration of all these different time scales. Another relevant measure of time is the mean escape time $\langle T\rangle$ for a trajectory to leave the neighborhood of an attracting set, and it will in general be different for the different attractors as can be seen in Fig. 3.

After establishing the complex features of the double rotor system, we proceed to show how this knowledge can help us in manipulating and controlling the behavior of this complex system. Unlike low-dimensional chaotic systems that are commonly controlled using the ideas introduced in Ref. [11], complex systems are not characterized by the existence of one large chaotic attractor but by the coexistence of many attractors. While the existence of a large chaotic attractor is critical to those control schemes [11], we argue that for a complex system, the unstable chaotic sets in the boundaries provide us with the necessary sensitivity and flexibility to gear the dynamics toward a specific periodic behavior using small perturbations. We can elect to stabilize an unstable periodic orbit embedded within a chaotic saddle in the boundary [12] or stabilize one of the (metastable) attracting sets as we show next.

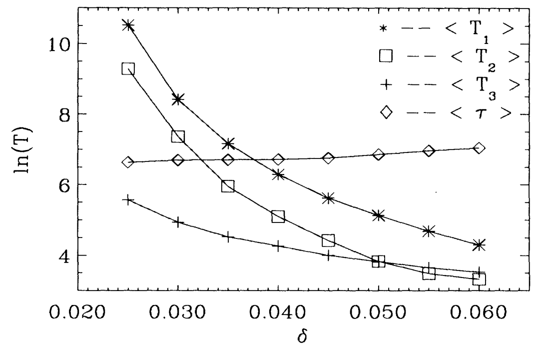

FIG. 3. The mean escape times $\langle T_i\rangle$ for some of the attractors and the average length of the chaotic transient $\langle\tau\rangle$ as a function of noise amplitude δ.

In controlling a metastable state in the complex double rotor map, we employ a simple feedback scheme. We denote the noiseless double rotor map as $\mathbf{x} \mapsto F(\mathbf{x})$ where $\mathbf{x}$ is the four-dimensional phase space coordinate, and the noisy double rotor map as $\tilde{F}(\mathbf{x}) \equiv F(\mathbf{x}) + \boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is the noise vector whose norm is bounded by δ. For simplicity, we assume the periodic orbit to be controlled is a fixed point (generalization to higher periodic orbit is fairly straightforward). If we labeled this fixed point as $\mathbf{x}_\star$, then in a neighborhood of it, we have the following linearization $F(\mathbf{x}_\star + \boldsymbol{\varepsilon}) = \mathbf{x}_\star + DF(\mathbf{x}_\star)\boldsymbol{\varepsilon}$, where the eigenvalues of $DF(\mathbf{x}_\star)$ are inside the unit circle (since $\mathbf{x}_\star$ is stable without noise). Suppose now that on the ith iterate, the trajectory lands in a neighborhood of this fixed point, so $\mathbf{x}_i = \mathbf{x}_\star + \boldsymbol{\varepsilon}$. Without control, $\mathbf{x}_i \mapsto \mathbf{x}_{i+1} = \tilde{F}(\mathbf{x}_i)$. However, assuming the linearization holds approximately for the noisy map near $\mathbf{x}_\star$, we can stabilize the fixed point with the addition of a controlling term, or $\hat{\mathbf{x}}_{i+1} = \mathbf{x}_{i+1} - DF(\mathbf{x}_\star)\boldsymbol{\varepsilon} = \tilde{F}(\mathbf{x}_i) - DF(\mathbf{x}_\star)(\mathbf{x}_i - \mathbf{x}_\star)$. Thus, the noisy trajectory with the above perturbation approaches the fixed point in due course. Since we want to achieve control using only small perturbations, the correction $|DF(\mathbf{x}_\star)\boldsymbol{\varepsilon}|$ is scaled when necessary so it will not exceed some predetermined upper bound of our choice.

Applying the correction to the noisy double rotor map, we control the dynamics of the system. In Fig. 4, we follow a typical noisy trajectory until it lands in a neighborhood of the desired metastable attractor we wish to control, then we turn on the control and let the system evolve for another thousand iterates in the neighborhood of the desired attractor. Control is then turned off, and we let the trajectory wander until it falls near the next desired metastable attractor. In this fashion, we stabilize, say, eight of the metastable attractors in the order we desire as demonstrated in Fig. 4. Furthermore, if the trajectory is caught in the vicinity of an attracting set that is undesirable, we can destabilize it by applying a small amount of noise. In fact, this was done in a brain experiment to control epilepsy [13] and a fluidized bed experiment to control slugging [14].

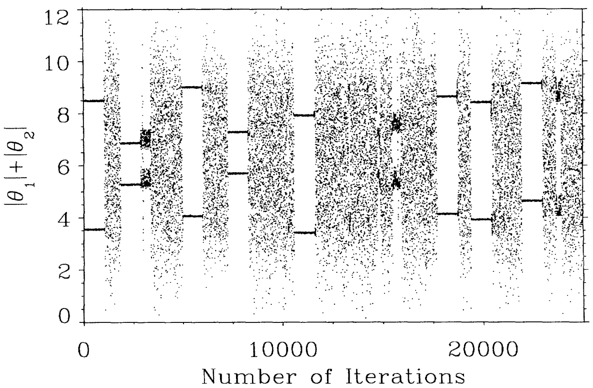

FIG. 4. Time series showing the result of applying the simple feedback control scheme to successively control eight different metastable states.

These two experiments demonstrate the general result that small perturbations, if chosen judiciously, not only can affect a desired outcome in these systems, but it also validates and extends these ideas to other systems.

In conclusion, we see that the myriad of possible behaviors in a complex system is of great utility if we are able to harness it. In fact, the ability of a complex system to access many different states, combined with its sensitivity, offers great flexibility in manipulating and controlling its dynamics. In general, many complex systems' behavior can be modified to suit our needs using only small perturbation strategies provided we are able to exploit their sensitivity.

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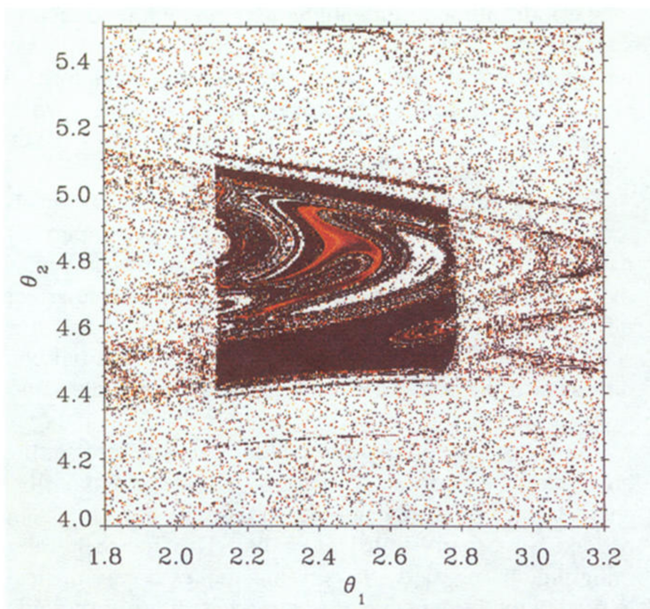

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