Geometric Model of the Generations

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(Received 15 August 1995)

We propose a geometric theory of flavor based on the discrete group $(S_3)^3$, in the context of the minimal supersymmetric standard model. The group treats three objects symmetrically, while making fundamental distinctions between the generations. The top quark is the only heavy quark in the symmetry limit, and the first and second generation squarks are degenerate. The hierarchical nature of Yukawa matrices is a consequence of a sequential breaking of $(S_3)^3$.

PACS numbers: 11.30.Hv, 12.15.Ff, 12.60.Jv

The smallness of the electroweak symmetry breaking scale and the hierarchical nature of the Yukawa matrices provide two of the most important problems of particle physics. Weak scale supersymmetry may well play a crucial role in the former, since it is the only symmetry that can protect the mass of an elementary scalar. However, weak scale supersymmetry widens the scope of flavor physics: any supersymmetric extension of the standard model possesses eleven flavor matrices rather than the three Yukawa matrices of the standard model. The additional eight flavor matrices all involve couplings to squarks and sleptons, and have therefore not been directly probed experimentally. However, rare processes such as the K_L-K_S mass difference provide experimental constraints on these flavor mixing matrices [1]. Hence the problem of flavor symmetries is greatly affected by the inclusion of weak scale supersymmetry.

It is frequently remarked that the most striking feature of the observed flavor physics is that the top quark is the only fermion with a mass of order the weak scale. In the context of the standard model this implies that only one entry of the three Yukawa matrices is of order unity, while all other entries are numerically small. In the context of the supersymmetric standard model, we find that there are now *two* features of flavor physics that must be considered at the zeroth order level: (1) the large mass of the top quark and (2) the near absence of flavor-changing neutral currents strongly suggests that scalars of a given charge of the light two generations are degenerate [1,2]. In this paper we explore the consequences of assuming that both of these salient features arise from a common origin—a flavor symmetry group G_f.

The existence of an exact flavor symmetry group at high energies is very plausible—it is suggested by the replication of generations. However, in many supersymmetric theories it becomes a necessity. Presumably the ultimate theory of flavor will involve no small parameters: all the dimensionless couplings will be of order unity and small mass ratios will result from hierarchies of dynamically generated mass scales, or perhaps from loop factors. If the supersymmetry breaking squark masses appear as hard interactions in such theories, as they do in supergravity models, then the couplings of order unity will lead to large radiative contributions to the squark masses [4]. The degeneracy between the first two generation scalars can only be maintained if the dimensionless couplings of the theory possess a non-Abelian flavor symmetry G_f.

What should we take for G_f? In the context of supergravity theories it was suggested that a U(N) invariance of the Kähler potential, where N is the total number of chiral superfluids of the theory, be used to protect the squark degeneracy [5]. However, in this paper we require that G_f also acts on the superpotential interactions that generate the fermion masses, so this U(N) invariance is not possible. Flavor symmetries that have been considered to date fall into two categories.

(1) Unified.—The group is such that in the symmetry limit there is no distinction whatever between generations. This occurs if the three generations are assigned to an irreducible representation that has three indistinguishable components—such as a triplet of SU(3).

(2) Asymmetric.—The action of the group is such that there is no symmetrical treatment of N objects, where $N = 3$ is the number of generations. There are many examples with G_f taken to be U(1)^n [6], SU(2) [7], or O(2) [8].

A unified G_f has the advantage of providing a more complete theory of flavor, whereas an asymmetric G_f does not provide an understanding of the difference between the generations. On the other hand, a unified G_f must be broken by couplings of order unity to obtain m_t, whereas an asymmetric G_f, such as SU(2), can provide an understanding of the salient flavor features even in the absence of symmetry breaking. In this paper we propose to combine the advantages of a unified G_f with those of an asymmetric G_f by introducing a third category of flavor symmetry.

(3) Symmetric.—The group has an action that is identical on three objects, yet has a representation structure that treats the generations differently.

In searching for such a group we are guided by three principles. (a) The fields of the theory are those of the minimal supersymmetric standard model: three generations and two Higgs doublets H_u and H_d. (b) The

0031-9007/95/75(22)/3985(4)$06.00

group should be a local discrete symmetry [9]. Continuous global symmetries are broken by quantum gravity [10] and should therefore be gauged. However, flavor symmetries must be broken to generate Yukawa matrices, and the breakdown of gauged flavor symmetries splits masses of different families due the D-term contribution [11]. (c) The representation structure of the three generations should be $(1 + 2)$, such that, in the G_f symmetry limit, the top quark Yukawa coupling is allowed, and the non-Abelian nature of the group maintains degeneracy between the scalars of the lighter two generations.

The discrete non-Abelian group with fewest group elements is the symmetric group S_3. By its very definition it acts symmetrically on three objects. Remarkably it has two singlets and a doublet as irreducible representations, and therefore offers an excellent match to the flavor problem of supersymmetric theories. The action of S_3 has a geometrical interpretation as all possible rotations in three dimensions that leave an equilateral triangle invariant. The three vectors representing the vertices of the triangle, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ in Fig. 1, are treated identically by the group. Yet the sums and differences of these vectors form a single representation ($\boldsymbol{v}_3$) and a doublet representation ($\boldsymbol{v}_1, \boldsymbol{v}_2$), whose two components have different group properties. For instance, an expectation value of $\boldsymbol{v}_1$ leaves a $\boldsymbol{Z}_2$ subgroup generated by (12) element, while that of $\boldsymbol{v}_2$ breaks S_3 completely. Despite a geometrical symmetry amongst three objects, there is also a geometrical understanding of the differences between the generations.

The group S_3 has six elements,

$$S_3 = \{e, (12), (13), (23), (123), (132)\}, \quad (1)$$

where e is the identity element. The two elements (123) and (132) are 120° rotations of the triangle around the axis $\boldsymbol{v}_3 = (1,1,1)/\sqrt{3}$, which form the $\boldsymbol{Z}_3$ subgroup of even permutations in S_3. The (12), (13), and (23) elements rotate the triangle by 180° around one of its symmetry axes,

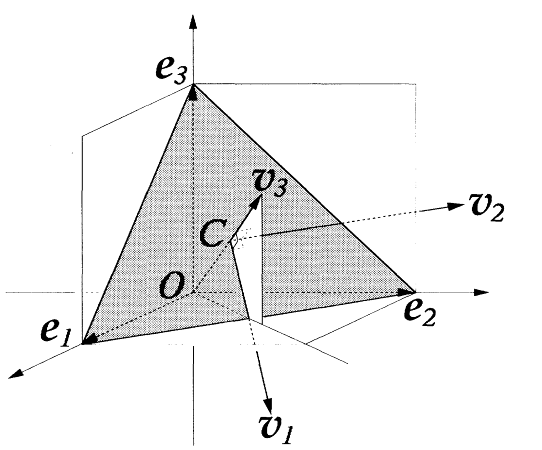

FIG. 1. S_3 acts as a rotation of the triangle spanned by three orthonormal vectors $\boldsymbol{e}_{1,2,3}$. The vector $\boldsymbol{v}_3$ corresponds to the $\mathbf{1}_A$ representation, and two vectors $\boldsymbol{v}_{1,2}$, in the plane of the triangle, to the $\mathbf{2}$ representation.

which are odd permutations. The vector $\boldsymbol{v}_3$ flips its sign under odd permutations but does not under even permutations. This is a nontrivial singlet representation that we call $\mathbf{1}_A$, and will be identified with the third generation later. Two other orthogonal vectors $\boldsymbol{v}_1 = (1,1,-2)/\sqrt{6}$ and $\boldsymbol{v}_2 = (-1,1,0)/\sqrt{2}$ form a doublet representation $\mathbf{2}$ of S_3. We identify them later with first and second generation fields, respectively. Any $\mathbf{2}$ representation can be written as a two-vector in $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ space. There are only three irreducible representations of S_3: $\mathbf{1}_A$ and $\mathbf{2}$ above and another singlet $\mathbf{1}_S$, which is a trivial representation (invariant). The $\mathbf{1}_S$ representation can be obtained as a symmetric product ${}^t\boldsymbol{x}\boldsymbol{y}$ of two $\mathbf{2}$'s, $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$, while $\mathbf{1}_A$ is an antisymmetric product ${}^t\boldsymbol{x}\sigma_2\boldsymbol{y}$. The other combinations form a $\mathbf{2}$ such as ${}^t\boldsymbol{x}\sigma_3\boldsymbol{y} \sim \boldsymbol{v}_1$ and ${}^t\boldsymbol{x}\sigma_1\boldsymbol{y} \sim -\boldsymbol{v}_2$ and $\mathbf{2}^3$ contains a totally symmetric invariant $({}^t\boldsymbol{x}\sigma_3\boldsymbol{y})z_1 - ({}^t\boldsymbol{x}\sigma_1\boldsymbol{y})z_2$. The decomposition of tensor products is shown in Table I.

Discrete non-Abelian groups with only singlet and doublet representations, such as S_3 [12] and Q_{2n} [13], have been used before to obtain a heavy top quark. However, this is clearly also possible with Abelian groups. In this paper we combine the supersymmetric motivation for some non-Abelian nature to G_f with the aesthetic desire for a symmetric flavor group.

Despite the encouraging features of S_3, it is not possible to satisfy the few guiding principles (a), (b), and (c) above using a single S_3 as G_f. For $\tilde{d}$ and $\tilde{s}$ to be degenerate, $(d,s)_L$ and $(d,s)_R$ should both transform as $\mathbf{2}$. Since $\mathbf{2} \times \mathbf{2} = \mathbf{1}_A + \mathbf{1}_S + \mathbf{2}$, both m_d and m_s are allowed by S_3, no matter whether H_d is assigned to $\mathbf{1}_A$ or to $\mathbf{1}_S$. An enlargement of the group is thus necessary. One possibility is to search for interesting structures in larger discrete groups, such as S_n, D_n, Q_{2n}, and $\Delta(3n^2)$ [13,14]. We find the geometric picture of the three generations arising from the symmetric action of S_3 to be sufficiently compelling that we prefer to replicate S_3 factors. Hence we consider a group $S_3^Q \times S_3^U \times S_3^D$ with each of Q, U, D transform as $\mathbf{1} + \mathbf{2}$ under its own S_3, while transforming trivially under other factors.

We identify the third generation with $\mathbf{1}_A$ rather than $\mathbf{1}_S$, because we would like to consider the discrete flavor group as an anomaly free gauge symmetry. The only anomaly one can discuss with the low-energy particle content alone is $S_3 \times H^2$ where H is either SU(2) or SU(3) in the standard model [15]. Consider the element (12), which leaves $\mathbf{1}_S$ and $\boldsymbol{v}_1$ in $\mathbf{2}$ invariant but changes sign of $\mathbf{1}_A$ and $\boldsymbol{v}_2$ in $\mathbf{2}$. To avoid an anomaly, the total number of $\mathbf{1}_A$ and $\mathbf{2}$ with a given quantum number has to be even. In our context, this requirement uniquely selects $\mathbf{1}_A + \mathbf{2}$.

TABLE I. Decomposition of tensor products of two representations into irreducible representations.

$\otimes$	$\mathbf{1}_S$	$\mathbf{1}_A$	$\mathbf{2}$
$\mathbf{1}_S$	$\mathbf{1}_S$	$\mathbf{1}_A$	$\mathbf{2}$
$\mathbf{1}_A$	$\mathbf{1}_A$	$\mathbf{1}_S$	$\mathbf{2}$
$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}_A \oplus \mathbf{1}_S \oplus \mathbf{2}$

The anomaly freedom of this choice can be easily understood by noting a vector $\mathbf{3}$ in an anomaly free group SO(3) decomposes to $\mathbf{1}_A + \mathbf{2}$. Furthermore, this choice is precisely the one that allows a geometric interpretation of families in terms of rotations. It is interesting to see that the three generations, although in a reducible representation $\mathbf{1}_A + \mathbf{2}$, require each other to render the theory consistent quantum mechanically.

The flavor transformation properties of the quarks are shown in Table II. The quantum number of H_u is fixed to allow m_t by G_f. Anomaly freedom then dictates an identical transformation for H_d. Because of this charge assignment, only the top quark is heavy in the $(S_3)^3$ symmetric limit. The top-bottom asymmetry, or more generally the up-down asymmetry, is built into the representation structure of the Higgs boson. On the other hand, squark mass matrices all have the form $\mathrm{diag}(M_1^2, M_1^2, M_3^2)$. The lepton sector will be discussed elsewhere.

Now we consider the breaking of $(S_3)^3$ symmetry and discuss its consequences on the Yukawa and squark mass matrices. In order to keep the number of breaking parameters as small as possible, we take the following "minimal" form of the Yukawa matrices [16],

$$Y_u = \begin{pmatrix} h_u & O(\sqrt{h_u h_c}) & -h_t\lambda^3 A(\rho + i\eta) \\ O(\sqrt{h_u h_c}) & h_c & -h_t\lambda^2 A \\ 0 & 0 & h_t \end{pmatrix}, \tag{2}$$

$$Y_d = \begin{pmatrix} h_d & h_s\lambda & 0 \\ O(h_s\lambda) & h_s & 0 \\ 0 & 0 & h_b \end{pmatrix}, \tag{3}$$

which correctly reproduce the Cabbibo-Kobayashi-Maskawa (CKM) matrix in Wolfenstein parametrization. The quark masses are related to the Yukawa couplings by $m_{u,c,t} = h_{u,c,t}\langle H_u\rangle$ and $m_{d,s,b} = h_{d,s,b}\langle H_d\rangle$. We assumed $(Y_u)_{12}$ and $(Y_u)_{21}$ to be $O(\sqrt{h_u h_c})$ and $(Y_d)_{12}$ to be $O(h_s\lambda)$, because larger off-diagonal elements need a fine-tuning in the determinant. We actually do not need these elements and can set them vanishing, but we keep them to make the discussion more general.

The largest breaking parameters in the Yukawa matrices are h_b, which transforms as $(\mathbf{1}_S, \mathbf{1}_A, \mathbf{1}_A)$, and $h_t\lambda^2 A$ as $(\mathbf{2}, \mathbf{1}_S, \mathbf{1}_S)$. h_b breaks $S_3^U \times S_3^D$ down to a subgroup $S_3^U \times S_3^D/\mathbf{Z}_2$, where (even, odd) and (odd, even) elements are removed. Note that the diagonal subgroup $S_3^{U,D}$ is a subgroup of the unbroken symmetry. $h_t\lambda^2 A$ is a $\boldsymbol{v}_1$ element in a doublet, and breaks S_3^Q to $S_2^Q \simeq \mathbf{Z}_2 = \{e, (12)\}$. This $\mathbf{Z}_2$ flips the sign of second generation and Higgs fields, while leaving the first generation field unchanged. Therefore Q_2 can acquire a Yukawa coupling while Q_1 cannot. h_c and h_s belong to breaking parameters $(\mathbf{2}, \mathbf{2}, \mathbf{1}_S)$ and $(\mathbf{2}, \mathbf{1}_A, \mathbf{2})$, respectively, and break the diagonal $S_3^{U,D}$ to $\mathbf{Z}_2$ as well, which still keeps all first generation fields massless. After including the smaller breaking parameters, the symmetry $(S_3)^3$ is completely broken. In this way, the hierarchical pattern of the Yukawa matrices can be obtained by a sequential breaking of the flavor symmetry.

TABLE II. Quantum number assignments of the fields under $(S_3)^3$ symmetry. Q refers to left-handed quark doublets, U (D) to right-handed up- (down-) type quarks

	Q	U	D	H_u	H_d
S_3^Q	$(\mathbf{1}_A, \mathbf{2})$	$\cdots$	$\cdots$	$\mathbf{1}_A$	$\mathbf{1}_A$
S_3^U	$\cdots$	$(\mathbf{1}_A, \mathbf{2})$	$\cdots$	$\mathbf{1}_A$	$\mathbf{1}_A$
S_3^D	$\cdots$	$\cdots$	$(\mathbf{1}_A, \mathbf{2})$	$\cdots$	$\cdots$

Now we turn to the squark mass matrices. Since the constraints from the flavor-changing neutral currents are at best of order a few times 10^{-3}, we work out the nondegeneracy in squark masses down to this order. It is straightforward to work out how the breaking parameters enter the scalar matrices. For m_Q^2 matrix, the leading correction comes from $(\mathbf{2}, \mathbf{1}_S, \mathbf{1}_S)$ with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ components of $O(h_t\lambda^2 A)$ and $O(h_t\lambda^3 A(\rho + i\eta))$, respectively. Therefore, m_Q^2 has the following form, which is Hermitian,

$$m_Q^2 \sim \begin{pmatrix} M_1^2 + m^2 h_t\lambda^2 A & \cdots & \cdots \\ m^2 h_t\lambda^3 A(\rho - i\eta) & M_1^2 - m^2 h_t\lambda^2 A & \cdots \\ m'^2 h_t\lambda^3 A(\rho - i\eta) & m'^2 h_t\lambda^2 A & M_3^2 \end{pmatrix}, \tag{4}$$

where a possible correction to $(m_Q^2)_{33}$ was absorbed into M_3^2. Here and hereafter, m^2 and m'^2 are arbitrary numbers comparable to M_1^2 and M_3^2, and they are in general different for Q, U, D. For the m_U^2 matrix, the only correction comes from the square of the $(\mathbf{2}, \mathbf{2}, \mathbf{1}_S)$ breaking parameter of $O(h_c^2)$. The resulting form is

$$m_U^2 \sim \begin{pmatrix} M_1^2 + m^2 h_c^2 & \cdots & \cdots \\ m^2\sqrt{h_c^3 h_u} & M_1^2 - m^2 h_c^2 & \cdots \\ m'^2\sqrt{h_c^3 h_u} & m'^2 h_c^2 & M_3^2 \end{pmatrix}. \tag{5}$$

The m_D^2 matrix receives corrections from two sources at the leading order. One is the square of the $(\mathbf{2}, \mathbf{1}_A, \mathbf{2})$ breaking parameter of $O(h_s^2)$, and the other is a product of three breaking parameters $(\mathbf{1}_S, \mathbf{1}_A, \mathbf{1}_A)$, $(\mathbf{2}, \mathbf{1}_A, \mathbf{2})$, and $(\mathbf{2}, \mathbf{1}_S, \mathbf{1}_S)$ of $O(h_s h_b h_t A\lambda^2)$. They are of the same order of magnitude and have the same group theoretical structure $(\mathbf{1}_S, \mathbf{1}_S, \mathbf{2})$. We keep only the first for simplicity and obtain

$$m_D^2 \sim \begin{pmatrix} M_1^2 + m^2 h_s^2 & \cdots & \cdots \\ m^2 h_s^2\lambda & M_1^2 - m^2 h_s^2 & \cdots \\ m'^2 h_s^2\lambda & m'^2 h_s^2 & M_3^2 \end{pmatrix}. \tag{6}$$

The authors of Ref. [3] listed the constraints on the off-diagonal mass matrix elements for $m_{\tilde{q}} \sim 1$ TeV in the basis where the Yukawa matrices are diagonal. We adopt their notation and list the constraints in Tables III, IV, and V. It is clear that our mass matrices satisfy all constraints rather easily. The constraints on the left-right mixing mass matrices, which are tightly constrained by S_3^3, are also easily satisfied.

A natural question is how much stronger the constraints become when we introduce further breaking parameters

TABLE III. The constraints and the consequence of $(S_3)^3$ symmetry on the mass splittings in $\tilde{d}$-$\tilde{s}$.

	$(\delta^d_{LL})_{12}$	$(\delta^d_{RR})_{12}$	$(\delta^d_{LR})_{12}$	$\langle\delta^d_{12}\rangle$
Upper bound [3]	0.05	0.05	0.008	0.006
This model	$h_t A\lambda^3$	$h_s^2\lambda$	$h_s^2\lambda$	$\sqrt{(\delta^d_{LL})_{12}(\delta^d_{RR})_{12}}$

TABLE IV. The constraints and the consequence of $(S_s)^3$ symmetry on the mass splittings in $\tilde{d}$-$\tilde{b}$.

	$(\delta^d_{LL})_{13}$	$(\delta^d_{RR})_{13}$	$(\delta^d_{LR})_{13}$	$\langle\delta^d_{13}\rangle$
Upper bound [3]	0.1	0.1	0.06	0.04
This model	$h_t A\lambda^2$	h_s^2	$h_b h_t A\lambda^3$	$\sqrt{(\delta^d_{LL})_{13}(\delta^d_{RR})_{13}}$

TABLE V. The constraints and the consequence of $(S_3)^3$ symmetry on the mass splittings in $\tilde{u}$-$\tilde{c}$. We assumed that the rotation angle between u and c is $O(\sqrt{h_u/h_c})$.

	$(\delta^u_{LL})_{12}$	$(\delta^u_{RR})_{12}$	$(\delta^u_{LR})_{12}$	$(\delta^u_{LR})_{12}$
Upper bound [3]	0.1	0.1	0.06	0.04
This model	$h_t A\lambda^3$	$\sqrt{h_u h_c^3}$	$\sqrt{h_u h_c^3}$	$\sqrt{(\delta^u_{LL})_{12}(\delta^u_{RR})_{12}}$

and introduce mixing in the right-handed fields as well. The off-diagonal elements of m_D^2 and m_U^2 can be much larger than the above estimates. However, they are at most of the same order as those in m_Q^2 if we assume a similar order of mixing angles in the right-handed fields. On the other hand, constraints become even weaker if we attribute all CKM angles to the down sector, since the breaking parameters are then proportional to h_b rather than h_t. A potentially dangerous breaking is that in $(\mathbf{1}_S, \mathbf{1}_S, \mathbf{1}_A)$ or $(\mathbf{1}_A, \mathbf{1}_S, \mathbf{1}_S)$, which do not contribute to the Yukawa matrices. However, they are presumably as small as h_u or h_d because they break the $\mathbf{Z}_2$ symmetry, which keeps the first generation fields massless.

In summary, we have proposed a geometric theory of flavor based on the discrete group $(S_3)^3$. The group acts symmetrically on three objects, yet gives fundamentally different characteristics to each generation. The three generations belong to a reducible representation $\mathbf{2} + \mathbf{1}_A$; although they are not unified, they require each other for anomaly cancellations. Only the top quark is heavy in the symmetry limit, and first- and second-generation squarks are degenerate. Hierarchical Yukawa matrices can be obtained by a sequential symmetry breaking. Flavor-changing processes are highly suppressed, allowing squarks at Fermilab Tevatron energies.

This work was supported in part by the U.S. Department of Energy under Contract No. DE-AC03-76SF00098 and by the National Science Foundation under Grant No. PHY-90-21139.

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