Gravitational Waves Generated by the Vacuum Stress

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We show that the $1/\Box$, $\Box \to -0$ asymptotic terms discovered in the one-loop vertices of quantum gravitating fields result in a new effect: a generation of the gravitational waves from the vacuum. This is an effect of the backreaction of the vacuum stress on the metric. A new method is proposed for calculating the energy of gravitational radiation that can be used in classical problems as well.

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The need of taking full account of the nonlinearity of the gravitational field equations when studying the radiation of isolated systems, even if this radiation is weak, has recently been stressed in the framework of classical theory [1]. Here we show that, in the framework of quantum theory, this nonlinearity results in a new mechanism of the energy emission: a generation of the gravitational waves from the vacuum. This is an effect of the backreaction of the vacuum stress on the metric. The backreaction of the vacuum on the course of the gravitational collapse is the main issue addressed in this and previous studies (see references below).

Our treatment is based on the expectation-value theory although the effect under consideration can also be discussed in terms of scattering theory. In the case of interest (e.g., in the collapse problem) the quantum fields are in an in-state [2] in which the matter quanta form a heavy classically behaved source of the gravitational field ($T^{\mu\nu}_{\rm source}$), and gravitons are in a coherent state so that, for the mean metric, they form a classical incoming gravitational wave. Here we consider the simplest case where there is no incoming gravitational radiation. The expectation-value equations for the gravitational field in such a state

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = 8\pi (T^{\mu\nu}_{\rm source} + T^{\mu\nu}_{\rm vac}) \qquad (1)$$

are obtained by varying the action $S = S_{\rm source} + S_{\rm vac}$, $S_{\rm vac} = S_1 + S_2 + S_3 + O[R^4]$, $S_1 = (1/16\pi) \int dx\, g^{1/2} R$,

$$S_2 = \frac{1}{2(4\pi)^2} \int dx\, g^{1/2} \sum_i \gamma_i(-\Box_2) \mathcal{R}_1 \mathcal{R}_2(i)\,, \qquad (2)$$

$$S_3 = \frac{1}{2(4\pi)^2} \int dx\, g^{1/2} \times \sum_i \Gamma_i(-\Box_1, -\Box_2, -\Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i)\,, \qquad (3)$$

$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, where $S_{\rm vac}$ is the action for the in-vacuum, and the rule is applied that, after the variation, all operator functions are given by the retarded boundary conditions [2,3]. With the operator functions in the spectral forms [2–4] the imposition of the boundary conditions boils down to replacing all inverse operators by the retarded Green functions. In (2) and (3), $\mathcal{R}_1\mathcal{R}_2(i)$ and $\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3(i)$ are nonlocal curvature invariants that form the bases of second order and third order, respectively. Examples of these invariants are $\mathcal{R}_1\mathcal{R}_2(1) = R_{1\mu\nu}R_2^{\mu\nu}$, $\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3(9) = R_1R_2R_3$, and the full bases can be found in [4]. The vacuum form factors $\gamma_i(-\Box)$ and $\Gamma_i(-\Box_1, -\Box_2, -\Box_3)$ have been calculated in the one-loop approximation for a generic quantum field model [3,4].

The rate of the energy radiation from an isolated system at the instant u of retarded time is given by the Bondi-Sachs equation [5], which is an exact consequence of the expectation-value equations (1):

$$-\frac{dM(u)}{du} = \frac{1}{4\pi} \int d^2S \left| \frac{\partial}{\partial u} C \right|^2 + \int d^2S\, \frac{1}{4} r^2 (T^{\mu\nu}_{\rm source} + T^{\mu\nu}_{\rm vac}) \nabla_\mu v \nabla_\nu v|_{\mathcal{I}^+}\,, \qquad (4)$$

$(\nabla u)^2 = 0$, $(\nabla u, \nabla r)|_{\mathcal{I}^+} = -1$, $(\nabla v)^2 = 0$, $(\nabla u, \nabla v)|_{\mathcal{I}^+} = -2$. Here $\partial C/\partial u$ is a complex function whose real and imaginary parts are the Bondi-Sachs news functions of the outgoing gravitational waves [5], r is the luminosity parameter along the light rays of the congruence $u =$ const, and the integrals are over the unit 2-sphere S. The last term in (4) is the flux of the vacuum energy through the future null infinity $\mathcal{I}^+$.

The behavior of $T^{\mu\nu}_{\rm vac}$ at $\mathcal{I}^+$ is determined by the behaviors of the form factors in the vacuum action (2) and (3) at the limit where one of the $\Box$ arguments tends to zero and the others are fixed [6]. The behaviors of all the form factors in one (each) of the arguments are generated by a single quantity $I^{\mu\nu}(\xi, x)$, which is a tensor function of the spacetime point x and a function of a parameter ξ defined by varying $\tilde{S}_{\rm vac} =$

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$S_{\rm vac} - S_1$ with respect to the Ricci tensor only: $\delta_R \tilde{S}_{\rm vac} = [1/2(4\pi)^2] \int dx\, g^{1/2} I^{\mu\nu}(-\Box, x) \delta R_{\mu\nu}(x)$. Here the notation δ_R points out that only the Ricci tensors in (2) and (3) are varied, and $I(-\Box, x)$ is $I(\xi, x)$ with ξ replaced by the operator $-\Box$ acting on x. We then have

$$T^{\mu\nu}_{\rm vac} = \frac{1}{2(4\pi)^2}(2\nabla^{(\mu}\nabla_\alpha I^{\nu)\alpha} - g^{\mu\nu}\nabla_\beta\nabla_\alpha I^{\alpha\beta} - \Box I^{\mu\nu}) + O[R_{..} \times I],\; I^{\mu\nu} = I^{\mu\nu}(-\Box_{\rm ret}, x), \tag{5}$$

and $I = I_2 + I_3 + O[R^3_{..}]$ where I_2 and I_3 are the contributions of the actions S_2 and S_3. The behaviors of these contributions at small ξ are [4]

$$I_2^{\mu\nu}(\xi, x) = O(\log \xi),$$
$$I_3^{\mu\nu}(\xi, x) = \xi^{-1} A^{\mu\nu}(x) + O(\log \xi). \tag{6}$$

The $O(\log \xi)$ behavior of $I(\xi, x)$ is what is needed for $T^{\mu\nu}_{\rm vac}$ to decrease at I^+ like $O(1/r^2)$, and the coefficient of the log ξ behavior determines the rate of the vacuum radiation in Eq. (4) [6]. The $1/\xi$ terms of $I(\xi, x)$ discovered [4] in the form factors of the third-order action (3) present a problem since they cause the $1/r$ decrease of $T^{\mu\nu}_{\rm vac}$ at I^+ and, therefore, apparently violate the asymptotic flatness of the expectation value of the metric. In scattering theory, the presence of these terms signals an infrared divergence of the on-shell amplitudes with gravitons.

The solution of this problem is as follows. It can be checked on the basis of the explicit expressions in [4] that, owing to certain constraints that exist among the coefficients of the $1/\Box$ asymptotic terms of the third-order form factors, the $A^{\mu\nu}(x)$ in (6) satisfies the condition

$$\nabla_\mu A^{\mu\nu}(x) + O[R^3_{..}] = 0 \tag{7}$$

and can, moreover, be brought to the form $A^{\mu\nu}(x) = -\nabla_\alpha\nabla_\beta K^{[\mu\alpha][\nu\beta]}(x) + O[R^3_{..}]$ in which the fulfillment of condition (7) is manifest. From (6) and the last equation in (5) we have

$$\nabla_\alpha I_3^{\alpha\nu} = -\frac{1}{\Box_{\rm ret}} \nabla_\alpha A^{\alpha\nu} + O[R^3_{..}] + O(1/r^2),$$

which, apart from the term proportional to $\Box$, is the only combination entering (5). Hence, thanks to (7), the $1/\xi$ terms of $I(\xi, x)$ cancel in the expression (5) for $T^{\mu\nu}_{\rm vac}$. The significance of these terms is that they determine the other contribution to the mass-loss formula (4): the news functions of the gravitational waves.

For the calculation of the news functions we use the fact that they appear as coefficients of the principle asymptotic terms of the Riemann tensor at null infinity:

$$\nabla_\alpha v \nabla_\mu v m_\beta m_\nu R^{\alpha\beta\mu\nu}|_{I^+} = -\frac{8}{r}\frac{\partial^2}{\partial u^2} C + O(1/r^2), \tag{8}$$

$(m, \nabla u) = (m, \nabla v) = (m, m) = 0$, $(m, m^*) = 2$. On the other hand, as proposed in [3], the Riemann tensor can be obtained by solving the differentiated Bianchi identities. The solution is expanded in powers of the Ricci tensor and, in the case of vanishing news functions at I^-, is expressed in terms of the retarded Green function,

$$R^{\alpha\beta\mu\nu} = \frac{2}{\Box_{\rm ret}}(\nabla^\alpha\nabla^{[\mu}R^{\nu]\beta} - \nabla^\beta\nabla^{[\mu}R^{\nu]\alpha} + O[R^2_{..}]). \tag{9}$$

By combining (8), (9), and the equations of motion (1) we find the vacuum contribution to the news functions

$$\frac{\partial}{\partial u} C_{\rm vac} = -\frac{\partial}{\partial u}\left(4\pi r m_\mu m_\nu \frac{1}{\Box_{\rm ret}} T^{\mu\nu}_{\rm vac}\right)\Big|_{I^+} + O[R^2_{..}]. \tag{10}$$

Upon insertion of expression (5) into (10) we obtain

$$\frac{\partial}{\partial u} C_{\rm vac} = \frac{\partial}{\partial u}\left[\frac{r}{8\pi} m_\mu m_\nu I^{\mu\nu}(-\Box_{\rm ret}, x)\right]\Big|_{I^+} + O[R^2_{..}],$$

whence it is seen that, at the leading approximation, $C_{\rm vac}$ hangs on the $1/\xi$, $\xi \to 0$ asymptotic terms of $I(\xi, x)$. Since there are no such terms in the form factors of the second-order action, the contribution of the third-order action [call it $C_{\rm vac}(3)$] is the lowest-order one,

$$\frac{\partial}{\partial u} C_{\rm vac}(3) = \frac{\partial}{\partial u}\left(\frac{r}{8\pi} m_\mu m_\nu \frac{1}{\Box_{\rm ret}} \nabla_\alpha\nabla_\beta K^{[\mu\alpha][\nu\beta]}\right)\Big|_{I^+}.$$

Its calculation involves only the coefficients of the asymptotic behaviors of the third-order form factors in (3): $\Gamma_i(-\Box_1, -\Box_2, -\Box_3) = (1/\Box_m) F_i^m(\Box_n, \Box_k) + O(\log(-\Box_m))$, $\Box_m \to -0$, $n < k$. By using the explicit form of the basis of third-order invariants [4], this calculation can be done for an arbitrary quantum-field model. Thus, for the one-loop contribution of spin-0 particles we obtain

$$K^{\mu\alpha\nu\beta} = 12F^1_{29}(\Box_1, \Box_2)[\nabla^\mu\nabla^\nu R_1^{\gamma\sigma} \cdot \nabla_\gamma\nabla_\sigma R_2^{\alpha\beta}] + 4F^3_{28}(\Box_1, \Box_2)[\nabla^\mu R_1^{\alpha\gamma} \cdot \nabla^\nu R^\beta_{2\gamma}]$$
$$+ 8F^3_{27}(\Box_1, \Box_2)[\nabla^{(\mu}R_1^{\nu)\gamma} \cdot \nabla_\gamma R_2^{\alpha\beta}] + 4F^1_{25}(\Box_1, \Box_2)[R_1^{\mu\beta} \cdot R_2^{\nu\alpha}]$$
$$+ [6(\Box_2 - \Box_1)F^1_{29}(\Box_1, \Box_2) + 8F^1_{27}(\Box_1, \Box_2) - 4F^3_{27}(\Box_1, \Box_2)][\nabla^\mu\nabla^\nu R_1^{\alpha\beta} \cdot R^2],$$

and the explicit form of the functions $F_i^m(\Box_1, \Box_2)$ is given in [4].

All massless particles including gravitons [7] contribute to the flux of $T^{\mu\nu}_{\text{vac}}$ through I^+. The energy of this component of radiation [the last term in (4)] is determined by the $\log(-\Box)$, $\Box \to -0$ terms of the vacuum form factors. On top of this, $T^{\mu\nu}_{\text{vac}}$ as a whole acts as a source of the gravitational field and causes a secondary radiation of gravitons. This component of radiation [the first term in (4)] has the shape of a classical wave but with a quantum amplitude, and its energy is determined by the $1/\Box$, $\Box \to -0$ terms of the vacuum form factors. An important difference between the two cases is that the gravitational-wave component will be nonvanishing only if the initial state has a sufficient asymmetry, whereas the contribution of gravitons in $T^{\mu\nu}_{\text{vac}}|_{I^+}$ is present even in a spherically symmetric in-state because the out-states in which these gravitons appear at I^+ are squeezed vacuum states rather than coherent states [7]. (For an estimate of the graviton emission rate from a spherically symmetric black hole see [8].)

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