Statistical Distributions of Level Widths and Conductance Peaks in Irregularly Shaped Quantum Dots

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Analytical expressions for width and conductance peak distributions for quantum dots with multichannel leads in the Coulomb blockade regime are presented for both limits of conserved and broken time-reversal symmetry. The results are valid for any number of nonequivalent and correlated channels, and the distributions are expressed in terms of the channel correlation matrix M in each lead. The matrix M is also given in closed form. A chaotic billiard is used as a model to test numerically the theoretical predictions.

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Advances in nanostructures technology make possible the manufacture of semiconductor devices known as quantum dots [1] where electrons are confined to very small two-dimensional regions. By connecting external leads to such devices, it is possible to study their electronic transport properties: the conductance can be measured as a function of the Fermi energy and/or as a function of a magnetic flux through the dot. Of particular interest are dots that are weakly coupled to leads, due to the presence of barriers at the interface between dot and leads. In these cases the resonance widths Γ are small compared to their mean spacing Δ. At low temperatures $kT < \Delta$, only one quasibound level participates in the conduction process. The resulting Coulomb blockade peaks of the conductance [2] are equally spaced, but their amplitude exhibits order of magnitude fluctuations.

For dot sizes that are smaller than the electron-impurity mean free path, conductance fluctuations are determined by the dot geometry. We discuss dot geometries which display classical chaotic motion. In such cases one can model their transport properties using the concepts of chaotic scattering [3]. Our results should also hold for weakly disordered dots in the quasi-zero-dimension limit. In Ref. [4] a statistical description of the conductance peaks in the Coulomb blockade regime was developed using the R-matrix formalism [5] and assuming that the dot's wave functions are described by random matrix theory (RMT). Conductance and decay width distributions were derived for one-channel leads, as well as for symmetric leads with several equivalent and uncorrelated channels. However, a more general theory would encompass leads with an arbitrary number of correlated and nonequivalent channels. Progress in this direction was made in Ref. [6], where by modeling leads as point contacts [7] spatial wave function correlations were taken into account. Using the supersymmetry method [8], exact formulas for the width and conductance peak distributions were obtained for two-point contact leads. However, the derivation was restricted to the case of broken time reversal symmetry, and could not be applied to leads with finite width or with more than two-point contacts. On the experimental side, we note that conductance distributions are becoming accessible; for ballistic open dots ($\Gamma \gg \Delta$) such distributions were recently measured [9], and similar experiments are underway in the Coulomb blockade regime.

In this Letter we present exact formulas for the width and conductance peak distributions for leads with any number of channels that are in general correlated and nonequivalent. These distributions are obtained both for conserved and broken time-reversal symmetry, and are completely characterized in terms of the channel correlation matrix M. Our results are valid for both the pointlike contacts and the continuous extended leads models. We are able to treat the most general case because our methods are based exclusively on RMT, which is technically simpler than the supersymmetry method used in Ref. [6]. To test our theory we use a chaotic billiard, the Africa [10], for which the statistical distributions of one-channel leads were recently studied in detail [11]. This model is particularly useful to study systems with strong channel correlations. In contrast, the correlations between nearest points in the discretized Anderson model of a disordered dot [6] were too weak to produce any significant change in the width distribution of two-point leads (as compared with the uncorrelated channels distribution). We remark that since the partial width is analogous to the wave function intensity [see Eq. (3) below], our results for the partial and total width distributions can be directly tested in the microwave cavity experiments [12], where the wave function intensities are measured at several points and are spatially correlated.

Provided that $\Gamma \ll kT < \Delta$, which is typical of experiments [1,2], the conductance peak amplitude for a two-lead geometry is given by [13]

$$G = \frac{e^2}{h} \frac{\pi}{2kT} g , \quad \text{where } g = \frac{\Gamma_\lambda^l \Gamma_\lambda^r}{\Gamma_\lambda^l + \Gamma_\lambda^r} , \tag{1}$$

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and Γ_λ^l (Γ_λ^r) is the partial decay width of a resonance λ into channels of the left (right) lead. Each lead can support $\Lambda^{l(r)}$ open channels so that $\Gamma_\lambda^{l(r)} = \sum_c |\gamma_{c\lambda}^{l(r)}|^2$, where $\gamma_{c\lambda}^{l(r)}$ is the partial amplitude to decay into channel c on the left (right). R-matrix theory gives [5]

$$\gamma_{c\lambda} = \left(\frac{\hbar^2 k_c P_c}{m}\right)^{1/2} \int dS \Phi_c^*(\mathbf{r}) \Psi_\lambda(\mathbf{r}), \qquad (2)$$

where Ψ_λ is the resonance wave function inside the dot (scattering region), Φ_c is the wave function of an open channel c in the lead (asymptotic region), and the integral is taken over the contact boundary between the lead and the dot. P_c and k_c are the channel penetration factor to tunnel through the barrier and the longitudinal wave number, respectively. An alternative modeling assumes that the quantum dot is connected to the leads by one or more pointlike contacts [7]. Every such point contact $\mathbf{r}_c$ is considered as a channel, and the corresponding partial amplitude is [6]

$$\gamma_{c\lambda} = (\alpha_c \mathcal{A} \Delta/\pi)^{1/2} \Psi_\lambda(\mathbf{r}_c), \qquad (3)$$

where $\mathcal{A}$ is the area of the dot, Δ is the mean resonance spacing, and α_c is the coupling parameter of the point contact to the dot. Expanding a resonance wave function with energy ε in a fixed basis of states with that energy $\Psi_\lambda(\mathbf{r}) = \sum_\lambda \psi_{\lambda\mu} \rho_\mu(\mathbf{r})$ (the sum is truncated to N terms, typically much larger than Λ), the partial width to decay to channel c can be expressed as a scalar product $\gamma_{c\lambda} \equiv \langle \boldsymbol{\phi}_c | \boldsymbol{\psi}_\lambda \rangle = \sum_\mu \phi_{c\mu}^* \psi_{\lambda\mu}$, where $\phi_{c\mu}^* \equiv (\hbar^2 k_c P_c/m)^{1/2} \int dS \Phi_c^*(\mathbf{r}) \rho_\mu(\mathbf{r})$ in the R-matrix formalism and $\phi_{c\mu}^* \equiv (\alpha_c \mathcal{A} \Delta/\pi)^{1/2} \rho_\mu^*(\mathbf{r}_c)$ in the pointlike contact model. Expressing the width as a scalar product allows us to treat the extended lead and the pointlike contact models in an equivalent manner.

The resonance states Ψ_λ are assumed to have Gaussian orthogonal ensemble–(GOE-) or Gaussian unitarity ensemble–(GUE-) like statistical properties, depending on the symmetry class to which the dynamics in the dot corresponds [4]. This assumption is valid both for dots with chaotic dynamics and for weakly disordered dots. The eigenvector components $(\psi_1, \psi_2, \ldots, \psi_N) \equiv \boldsymbol{\psi}$ (in the following we shall omit the eigenvector label λ) are therefore randomly distributed $P(\boldsymbol{\psi}) \propto \delta(\sum_{\mu=1}^N |\psi_\mu|^2 - 1)$ [14]. The joint distribution of the partial width amplitudes $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_\Lambda)$ for Λ channels is given by

$$P(\boldsymbol{\gamma}) \propto \int D[\boldsymbol{\psi}] \delta\left(\sum_{\mu=1}^N |\psi_\mu|^2 - 1\right) \prod_{c=1}^\Lambda \delta(\gamma_c - \langle \boldsymbol{\phi}_c | \boldsymbol{\psi} \rangle), \qquad (4)$$

where the metric is $D[\boldsymbol{\psi}] \equiv \prod_{\mu=1}^N d\psi_\mu$ for the GOE and $D[\boldsymbol{\psi}] = \prod_{\mu=1}^N d\psi_\mu^* \, d\psi_\mu / 2\pi i$ for the GUE. To evaluate (4) we transform $\boldsymbol{\phi}_c = \sum_{c'} \hat{\boldsymbol{\phi}}_{c'} F_{c'c}$ to obtain a new set of orthonormal channels $\langle \hat{\boldsymbol{\phi}}_c | \hat{\boldsymbol{\phi}}_{c'} \rangle = \delta_{cc'}$. In the limit $N \to \infty$, provided that $\Lambda \ll N$, one finds [15]

$$P(\boldsymbol{\gamma}) = (\det M)^{-\beta/2} e^{-\frac{\beta}{2} \boldsymbol{\gamma}^\dagger M^{-1} \boldsymbol{\gamma}}, \qquad (5)$$

where $M \equiv (N F^\dagger F)^{-1}$. The distribution (5) is normalized with the measure $D[\boldsymbol{\gamma}] \equiv \prod_{c=1}^\Lambda d\gamma_c / 2\pi$ for the GOE ($\beta = 1$) and $D[\boldsymbol{\gamma}] \equiv \prod_{c=1}^\Lambda d\lambda_c^* \, d\lambda_c / 2\pi i$ for the GUE ($\beta = 2$). Note that for both ensembles the joint partial width amplitudes distribution is Gaussian, and it follows that M is just the correlation matrix of the partial widths $M_{cc'} = \overline{\gamma_c^* \gamma_{c'}} = \langle \boldsymbol{\phi}_c | \boldsymbol{\phi}_{c'} \rangle / N$. In general, the channels are correlated (nonorthogonal) and nonequivalent, i.e., have different average partial widths.

Recalling Eqs. (2) and (3), the spatial autocorrelation function $C(\Delta\mathbf{r}) \equiv \overline{\Psi^*(\mathbf{r}) \Psi(\mathbf{r} + \Delta\mathbf{r})} / \overline{|\Psi(\mathbf{r})|^2}$ plays a central role in deriving explicit expressions for the correlation matrix M. For fully chaotic systems with time reversal symmetry, Berry obtained $C(\Delta\mathbf{r})$ semiclassically, assuming that classical orbits cover uniformly the energy surface [16]. For an eigenstate of a chaotic billiard with energy $\varepsilon = \hbar^2 k^2 / 2m$, $C(\Delta\mathbf{r}) = J_0(k|\Delta\mathbf{r}|)$. Alternatively, a resonance at energy ε can be expanded inside the dot in the fixed basis $\rho_\mu(\mathbf{r}) = \mathcal{A}^{-1/2} \exp(i\mathbf{k}_\mu \cdot \mathbf{r})$, where the μ's correspond to N different (random) orientations of $\mathbf{k}$. In the spirit of RMT, assuming that the expansion coefficients ψ_μ are Gaussian (this hypothesis was confirmed for the Africa billiard [17]) and using $\overline{\psi_\mu^* \psi_{\mu'}} = N^{-1} \delta_{\mu\mu'}$, one can rederive Berry's result; see also Ref. [18]. The presence of a small magnetic flux Φ introduces corrections to $C(\Delta\mathbf{r})$ which are small in the semiclassical limit and are of the order $\hbar^2 (\Phi/\Phi_0)^2 / 2m\mathcal{A}\varepsilon \ll 1$. Thus, the correlation matrix is given by

$$M_{cc'} = \frac{\hbar^2 (k_c P_c k_{c'} P_{c'})^{1/2}}{m\mathcal{A}} \int dS \int dS' \, \Phi_c^*(\mathbf{r}) \times J_0(k|\mathbf{r} - \mathbf{r}'|) \Phi_c(\mathbf{r}') \qquad (6)$$

for the finite width leads, and

$$M_{cc'} = \frac{\Delta (\alpha_c \alpha_{c'})^{1/2}}{\pi} J_0(k|\mathbf{r}_c - \mathbf{r}_c'|) \qquad (7)$$

for the point contact model.

We turn next to the calculation of the total width distribution $P(\Gamma)$ in a given lead that supports Λ channels and is characterized by a correlation matrix M. Using (5) and $\Gamma = \sum_c |\gamma_c|^2$, the characteristic function of $P(\Gamma)$ is readily obtained and we find

$$P(\Gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, \frac{e^{-it\Gamma}}{[\det(I - 2itM/\beta)]^{\beta/2}} \,. \qquad (8)$$

The matrix M is Hermitian and positive definite, so that its eigenvalues w_c^2 are all positive. Since Γ is invariant under orthogonal (unitary) transformations, $P(\Gamma)$ depends only on w_c^2 an Eq. (8) can be evaluated by contour integration. When all eigenvalues of M are nondegenerate, we find for the GUE case

$$P_{\mathrm{GUE}}(\Gamma) = \frac{1}{\prod_c w_c} \sum_{c=1}^\Lambda e^{-\Gamma/w_c^2} \left[\prod_{c' \neq c} \left(\frac{1}{w_{c'}^2} - \frac{1}{w_c^2} \right) \right]^{-1} . \qquad (9)$$

For two equivalent channels ($M_{11} = M_{22} = \overline{\Gamma}/2$), Eq. (9) coincides with the result of [6]

$$P_{\mathrm{GUE}}(\hat{\Gamma}) = \frac{2}{|f|} e^{-2\hat{\Gamma}/(1-|f|^2)} \sinh\left(\frac{2|f|}{1 - |f|^2} \hat{\Gamma} \right), \qquad (10)$$

where $\hat{\Gamma} = \Gamma/\overline{\Gamma}$, and $f = M_{12}/\sqrt{M_{11}M_{22}}$ measures the degree of correlation between the two channels.

For the GOE case, the integral in (8) is still straightforward but cannot be expressed in a simple form as above. Labeling the inverse eigenvalues of M in increasing order $w_1^{-2} < w_2^{-2} < \cdots$, the contour integral gives

$$P_{\mathrm{GOE}}(\Gamma) = \left(\pi 2^{\Lambda/2} \prod_c w_c\right)^{-1} \sum_{m=1} \int_{(2w_{2m-1}^2)^{-1}}^{(2w_{2m}^2)^{-1}} d\tau \, \frac{e^{-\Gamma\tau}}{\sqrt{\prod_{s=1}^{2m-1}(r - 1/2w_s^2) \prod_{s'=2m}^{\Lambda}(1/2w_{s'}^2 - \tau)}}, \tag{11}$$

where, for an odd number of channels Λ, we define $1/2w_{\Lambda+1}^2 \to \infty$. For the case of two equivalent but correlated channels, Eq. (11) reduces to

$$P_{\mathrm{GOE}}(\hat{\Gamma}) = \frac{1}{\sqrt{1 - f^2}} \, e^{-\hat{\Gamma}/(1-f^2)} I_0\left(\frac{f}{1 - f^2}\hat{\Gamma}\right), \tag{12}$$

where $f = M_{12}/\sqrt{M_{11}M_{22}}$ and I_0 is the modified Bessel function of order 0.

To test the RMT predictions, we modeled a quantum dot by the Africa billiard [10,11]. The shape of this billiard is determined by the image of the circle of radius one in the complex z plane under the conformal mapping $w(z) = (z + bz^2 + ce^{i\delta}z^3)/\sqrt{1 + 2b^2 + 3c^2}$. We studied the case $b = 0.2$, $c = 0.2$, and $\delta = \pi/2$ which has a classical fully chaotic phase space [10]. For the statistical study of the case of broken time-reversal symmetry, we consider the billiard threaded by an Aharonov-Bohm flux line $\Phi = \alpha\Phi_0$ [11,19]. We choose $\alpha = 1/4$ where the resonance fluctuations are known to be GUE-like [11]. The study considers 50 eigenstates starting from the 100th.

To investigate the eigenfunction amplitude correlations $C(\Delta\mathbf{r})$, we have used the eigenfunctions of the billiard with von Neumann boundary conditions. The results are shown in the inset to Fig. 1 where the correlations in the model (solid line) compare well with the theoretical result $J_0(k\Delta r)$ (dashed line). In what follows, we assume for simplicity that the penetration factors k_cP_c are channel independent. We first studied extended leads by taking the contact region of the lead and the dot to have a finite width D. In this case, the channels are defined by the allowed transverse momenta $\pi c/D$ ($c = 1, \ldots, \Lambda$) where $\Lambda = \mathrm{int}[kD/\pi]$. To guarantee that the correlation matrix M is the same for different eigenfunctions of the billiard, we choose D such that kD is constant and scale the partial amplitude (2) by k. We find that the channels in this case are weakly correlated. Thus, the total width distribution is similar to the case of uncorrelated equivalent channels, which gives a χ^2 distribution with Λ (GOE) or 2Λ (GUE) degrees of freedom. However, this changes if the barrier penetration factors have a strong energy dependence. Our model calculations agree nicely with the RMT predictions (see Fig. 1).

We also studied the model of leads with Λ pointlike contacts by choosing for each lead a sequence of Λ equally spaced points on the Africa boundary. Thus, according to (7), M is completely determined by Λ and $k|\Delta\mathbf{r}|$ (where $|\Delta\mathbf{r}|$ is the distance between two neighboring points). In Fig. 2 we compare the results of this model (histograms) with the theoretical predictions (solid lines) for $\Lambda = 4$ and different values of $k|\Delta\mathbf{r}|$.

To calculate the conductance distribution, which is the measurable quantity for quantum dots, we assume that the

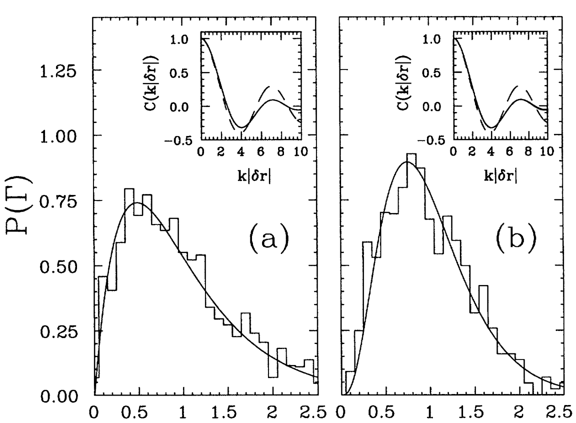

FIG. 1. Total width distributions $P(\Gamma)$ for extended leads with $\Lambda = 4$. Γ is measured in units of its average value. The solid lines are the theoretical distributions, and the histograms are results from the Africa billiard for the (a) GOE and (b) GUE cases (see text). The inset shows the spatial wave function correlation $C(\Delta\mathbf{r})$ calculated for the Africa (solid line) compared with $J_0(k|\Delta\mathbf{r}|)$ dashed line.

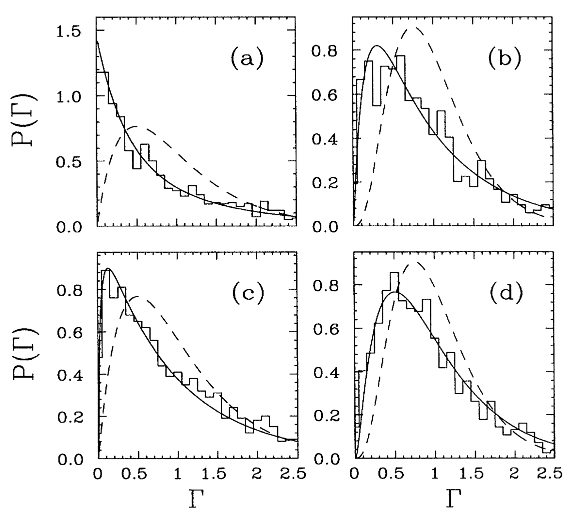

FIG. 2. Total width distributions $P(\Gamma)$ for $\Lambda = 4$ point-contact leads. (a) GOE, $k|\Delta\mathbf{r}| = 0.5$; (b) GUE, $k|\Delta\mathbf{r}| = 0.5$; (c) GOE, $k|\Delta\mathbf{r}| = 1$; and (d) GUE, $k|\Delta\mathbf{r}| = 1$. The solid lines correspond to the theoretical distributions, the dashed lines to uncorrelated channels, and the histograms to the Africa billiard calculations.

left and right leads are uncorrelated and characterized by M^l and M^r, respectively. We then use $P(g) = \int d\Gamma^l\, d\Gamma^r\, \delta(g - \Gamma^l\Gamma^r/(\Gamma^l + \Gamma^r))P(\Gamma^l)P(\Gamma^r)$, where $P(\Gamma)$ is given by (9) or (11). In the absence of time-reversal symmetry we find

$$P_{\mathrm{GUE}}(g) = \frac{16g}{(\prod_c v_c \prod_d w_d)^2} \sum_{c,d} e^{-(1/v_c^2+1/w_d^2)g} \left[\prod_{c'\neq c} \left(\frac{1}{v_{c'}^2} - \frac{1}{v_c^2} \right) \prod_{d'\neq d} \left(\frac{1}{w_{d'}^2} - \frac{1}{w_d^2} \right) \right]^{-1} \times \left[K_0\left(\frac{2g}{v_c w_d}\right) + \frac{1}{2}\left(\frac{v_c}{w_d} + \frac{w_d}{v_c}\right) K_1\left(\frac{2g}{v_c w_d}\right) \right], \tag{13}$$

where v_c (w_d) are the eigenvalues of M^l (M^r), and K_0 (K_1) are the modified Bessel functions of order 1 (0).

The result of [4,8] is a special case of (13) for one channel lead with $\overline{\Gamma}^l = \overline{\Gamma}^r$ (i.e., $v_1 = w_1$), while the distribution of Ref. [6] is obtained for two (equivalent) channels leads. For time reversal symmetric systems, we also obtained a closed formula for $P(g)$, which has similar structure to (13). Figure 3 shows a comparison between the theoretical conductance distributions and those calculated for the Africa billiard for Λ-point symmetric leads with $k|\Delta\mathbf{r}| = 1$ and for various values of Λ.

In conclusion, we have derived in closed form the width and conductance peak distributions in a quantum dot, for leads with any number of correlated and/or nonequivalent channels, and in the presence or absence of time reversal symmetry. The only required input to determine the distributions is the channel correlation matrix M, for which an explicit expression was obtained. Our results for the decay widths could also be applied to compound nucleus reactions in the limit of isolated resonances, where M is evaluated by the optical model.

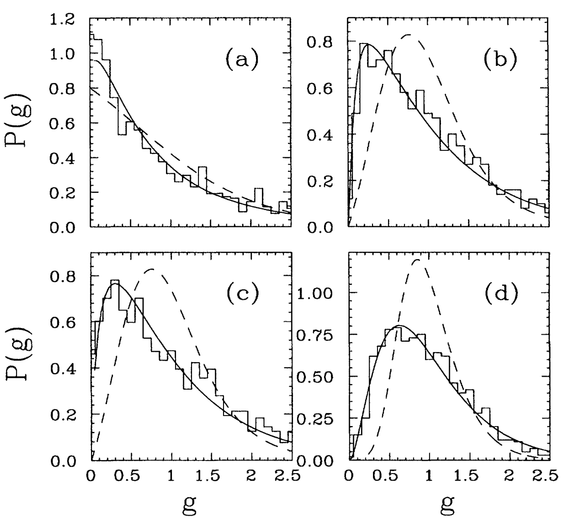

FIG. 3. Conductance peak distributions $P(g)$ for Λ point-contact symmetric leads ($M^l = M^r$) with $k|\Delta\mathbf{r}| = 1$. (a) GOE, $\Lambda = 2$; (b) GUE, $\Lambda = 2$; (c) GOE, $\Lambda = 4$; and (d) GUE, $\Lambda = 4$. The convention for the lines is as in Fig. 2.

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