Anomalous Scaling of the Passive Scalar

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We establish anomalous inertial range scaling of the structure functions for a model of homogeneous, isotropic advection of a passive scalar by a random velocity field. The velocity statistics is taken as Gaussian with decorrelation in time and velocity differences scaling as $|\mathbf{x}|^{\kappa/2}$ in space, with $0 \leq \kappa < 2$. The scalar is driven by a random forcing acting on spatial scale L. The structure functions for the scalar are well defined as the diffusivity is taken to zero and acquire anomalous scaling for large L. The anomalous exponent is calculated explicitly for the fourth structure function and for small κ.

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In 1941 A. N. Kolmogorov [1] argued that in fully developed turbulence there exists a range of scales where the velocity structure functions are independent of the cutoffs provided by the scales of energy pumping and dissipation. Ever since, a debate has been going on as to whether there are corrections to the scaling exponents predicted by Kolmogorov and whether such corrections depend on the dissipation or the pumping scale or both.

In this Letter we consider a similar question in a simpler model of turbulent phenomena. The model, which has attracted much attention recently [2–7], describes the passive advection in a random velocity field $\mathbf{v}(t, \mathbf{x})$ of a scalar quantity T. The density $T(t, \mathbf{x})$ satisfies the equation

$$\partial_t T + (\mathbf{v} \cdot \nabla) T - \nu \Delta T = f, \qquad (1)$$

where ν denotes the molecular diffusivity of the scalar T and $f(t, \mathbf{x})$ is an external source driving the system. We take $\mathbf{v}(t, \mathbf{x})$ and $f(t, \mathbf{x})$ to be mutually independent Gaussian random fields with zero mean and covariances

$$\langle v^i(t, \mathbf{x}) v^j(t', \mathbf{x}') \rangle = \delta(t - t') D^{ij}(\mathbf{x} - \mathbf{x}'), \qquad (2)$$

$$\langle f(t, \mathbf{x}) f(t', \mathbf{x}') \rangle = \delta(t - t') C\left(\frac{\mathbf{x} - \mathbf{x}'}{L}\right) \equiv \delta(t - t') C_L(\mathbf{x} - \mathbf{x}'). \qquad (3)$$

The forcing covariance C is assumed to be a real, smooth, rotationally invariant, positive-definite function with rapid decay at spatial infinity so that the forcing is homogeneous, isotropic, and takes place on the ("integral") scale L.

The velocity covariance D is taken to mimic the situation in a real turbulent flow with the structure function $\langle [\mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, 0)]^2 \rangle$ proportional to $|\mathbf{x}|^\kappa$ for $\kappa > 0$. Concretely, we set

$$D^{ij}(\mathbf{x}) = D_0 \int e^{i\mathbf{k}\cdot\mathbf{x}} (\mathbf{k}^2 + m^2)^{-(3+\kappa)/2} \times (\delta^{ij} - k^i k^j / \mathbf{k}^2) \frac{d^3\mathbf{k}}{(2\pi)^3}, \qquad (4)$$

where the transverse projector ensures the incompressibility of $\mathbf{v}$. The small m^2 is an infrared cutoff making the integral convergent for $0 < \kappa < 2$. Writing $D(\mathbf{x}) = D(0) - \tilde{D}(\mathbf{x})$, we have $D^{ij}(0) \propto D_0 m^{-\kappa}$ which diverges with $m \to 0$, but the velocity structure function has a limit

$$\lim_{m \to 0} \tilde{D}^{ij}(\mathbf{x}) = D_1 [(2 + \kappa) \delta^{ij} |\mathbf{x}|^\kappa - \kappa x^i x^j |\mathbf{x}|^{\kappa - 2}] \qquad (5)$$

which is a homogeneous function of $\mathbf{x}$.

$$D_1 \equiv \frac{\Gamma((2 - \kappa)/2)}{2^{2+\kappa} \pi^{3/2} \kappa (3 + \kappa) \Gamma((3 + \kappa)/2)} D_0$$

and both constants have dimension $(\text{length})^{2-\kappa}/\text{time}$.

We would like to study the statistical properties of the solutions of Eq. (1) in the regime of small ν, small m (which may be viewed as the inverse of another integral scale), and large L. In particular, the universality question for the passive scalar may be formulated as inquiring about the existence of the limit of the correlation functions $\langle \prod_n T(t_n, \mathbf{x}_n) \rangle$ in a stationary state of the system when $\nu, m, L^{-1} \to 0$ and about the independence of such a limit of the shape of the source covariance C. We will show that the model possesses an "inertial" range of scales $(\nu / D_1)^{1/\kappa} \ll |\mathbf{x}| \ll \min(L, m^{-1})$ where these correlators become independent of ν, have a limit as $\nu \to 0$ and $m \to 0$ (independent of the order), but in general have *nonuniversal* (i.e., dependent on the forcing covariance) contributions involving positive powers of L. In particular we show that the structure functions

$$S_{2N}(\mathbf{x}) \equiv \langle [T(\mathbf{x}) - T(0)]^{2N} \rangle \sim \gamma_{2N} (L / |\mathbf{x}|)^{\rho_{2N}} |\mathbf{x}|^{(2-\kappa)N} \qquad (6)$$

for $|\mathbf{x}| \ll L$ in the $\nu, m = 0$ limit. The amplitudes γ_{2N} are κ and C dependent and the anomalous exponents ρ_{2N} depend on κ but not on C. We find that $\rho_2 = 0$ but

$$\rho_4 = \tfrac{4}{5} \kappa + \mathcal{O}(\kappa^2) \qquad (7)$$

for κ small. [A similar calculation in general space dimension $d > 2$ gives $\rho_4 = 4/(d + 2)\kappa + \mathcal{O}(\kappa^2)$, which is consistent with the $1/d$ analysis of the final version of [5].] The Hölder inequality implies that ρ_N is a convex function of N. It follows that all ρ_{2N} for $N = 2, 3, \ldots$

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are strictly positive $[\rho_{2N} \geq (N-1)\rho_4]$ and that they increase with N. Thus structure functions of order 4 and higher exhibit anomalous scaling and have explicit integral scale dependence. The value (7) differs from the prediction of [2] which gives $\rho_4 = 1 - \frac{5}{7}\kappa + \mathcal{O}(\kappa^2)$.

It should be stressed that our results have not been obtained by a perturbation expansion in powers of κ applied directly to $S_{2N}(\mathbf{x})$. This would be a wrong strategy: Physically, the $\kappa \to 0$ limit corresponds to a purely diffusive regime rather than to a flux phase appearing for $\kappa > 0$; mathematically, the $\kappa \to 0$ limit does not commute with the $L \to \infty$ one. Instead, in a renormalization group spirit, we apply the perturbation expansion in κ to a properly identified single scale problem where it is under full control.

In the stationary state the scalar correlations satisfy linear partial differential equations (PDE's) [8,9]. In the presence of the UV and IR cutoffs ν and m, L they have well defined representations in terms of the Green functions of the corresponding differential operators which we now recall (for more details, see [10,11]). Suppressing the spatial variable, the solution of Eq. (1) with the initial condition T_0 at $t = t_0$ takes the form

$$T(t) = R(t, t_0)T_0 + \int_{t_0}^{t} R(t, s)f(s)\,ds, \qquad (8)$$

where $R(t, t_0)$ is given by the time ordered exponential $(t \geq t_0)$

$$R(t, t_0) = \mathcal{T} e^{\int_{t_0}^{t}[\nu\Delta + \mathbf{v}(\tau)\cdot\nabla]d\tau}. \qquad (9)$$

Thus to calculate the correlations of T we need to evaluate expectations of products of matrix elements of $R(t, t_0)$. Using the tensor product notation $R(t, t_0)^{\otimes N}$ as a bookkeeping device for all such products, one obtains

$$\langle R(t, t_0)^{\otimes N}\rangle = e^{-(t-t_0)\mathcal{M}_N}, \qquad (10)$$

where $\mathcal{M}_N$ is the differential operator

$$\mathcal{M}_N = -\sum_{n=1}^{N}[\nu\Delta_{\mathbf{x}_n} + \tfrac{1}{2}\mathcal{D}^{ij}(0)\partial_{x_n^i}\partial_{x_n^j}] - \sum_{n<n'}\mathcal{D}^{ij}(\mathbf{x}_n - \mathbf{x}_{n'})\partial_{x_n^i}\partial_{x_{n'}^j}. \qquad (11)$$

The Gaussian integral of the time ordered exponentials is calculable due to the independence of $\mathbf{v}$'s at different times. The $\frac{1}{2}\mathcal{D}^{ij}(0)$ term is the contribution of contractions within a single R and the last terms come from contractions between different R's.

To get hold of the steady state of the scalar, let us first consider the two-point function. From (8), we obtain

$$\langle T(t)^{\otimes 2}\rangle = e^{-t(t-t_0)\mathcal{M}_2}T(t_0)^{\otimes 2} + \int_{t_0}^{t} ds\, e^{-(t-s)\mathcal{M}_2}C_L. \qquad (12)$$

When $t_0 \to -\infty$, the term with $T(t_0)$ disappears due to the positivity of $\mathcal{M}_2$ and we obtain for the steady state

$$\langle T^{\otimes 2}\rangle = \mathcal{M}_2^{-1}C_L. \qquad (13)$$

Because of the translational invariance of C, $D^{ij}(0)$ (divergent when $m \to 0$) will not contribute to (13): $\mathcal{M}_2$ commutes with (three-dimensional) translations and in the action on translation-invariant functions of $\mathbf{x}_1 - \mathbf{x}_2 \equiv \mathbf{x}$ reduces to

$$\mathcal{M}_2 = -2\nu\Delta - \tilde{\mathcal{D}}^{ij}(\mathbf{x})\partial_i\partial_j. \qquad (14)$$

Since $\tilde{\mathcal{D}}^{ij}(\mathbf{x}) \equiv \mathcal{D}^{ij}(0) - \mathcal{D}^{ij}(\mathbf{x})$ has an $m \to 0$ limit given by (5), so does the operator $\mathcal{M}_2$ in the action on translation-invariant functions and when $\nu \to 0$ it becomes a singular elliptic operator $\mathcal{M}_2^{\mathrm{sc}} = -D_1[(2+\kappa)\delta^{ij}|\mathbf{x}|^{\kappa} - \kappa x^i x^j|\mathbf{x}|^{\kappa-2}]\partial_i\partial_j$. It is now easy to analyze (13) as the various cutoffs ν, m, L are removed using the rotational invariance of $\mathcal{M}_2$. In the $m \to 0$ and $\nu \to 0$ limits (which commute, we could also take m proportional to L^{-1} with no loss), one obtains for the two-point function $F_2(|\mathbf{x}|) \equiv \langle T(\mathbf{x})T(0)\rangle$

$$F_2(r) = \gamma_2 L^{2-\kappa} - \frac{2\epsilon}{3(2-\kappa)D_1}r^{2-\kappa}\left[1 + \mathcal{O}\left(\frac{r^2}{L^2}\right)\right], \qquad (15)$$

where γ_2 is a nonuniversal (i.e., C-dependent) constant and $\epsilon = \frac{1}{2}C(0)$ is the energy dissipation rate of the scalar. Note that the nonuniversal term (a constant) is annihilated by $\mathcal{M}_2^{\mathrm{sc}}$. This has to be so if the equation $\mathcal{M}_2^{\mathrm{sc}}F_2 = C_L$ is to be satisfied: The right-hand side (RHS) becomes universal in the limit $L \to \infty$ so all nonuniversal terms in $F_2(r)$ surviving in this limit have to be annihilated by $\mathcal{M}_2^{\mathrm{sc}}$. A similar mechanism will work for higher point functions. The constant term of F_2 drops out from the second structure function which has a universal $L \to \infty$ limit so that the exponent $\rho_2 = 0$. The same universal result holds approximately in the whole inertial range $\eta \ll r \ll \min(L, m^{-1})$, where the Kolmogorov scale $\eta = (\nu/D_1)^{1/\kappa}$.

Let us now analyze the higher point correlators. Proceeding as with the two-point function, the steady state solution in terms of the operators $\mathcal{M}_N$ follows after some simple algebra. For the four-point function one gets

$$\left\langle\prod_{n=1}^{4}T(\mathbf{x}_n)\right\rangle = F_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) + F_4(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_4) + F_4(\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2, \mathbf{x}_3) \qquad (16)$$

with the single channel function

$$F_4 = \mathcal{M}_4^{-1}(\mathcal{M}_2^{-1} \otimes 1 + 1 \otimes \mathcal{M}_2^{-1})C_L \otimes C_L. \qquad (17)$$

Expression (17) and its higher point counterparts [11] involving the Green functions of higher $\mathcal{M}_{2N}$'s are well defined for ν, m, L^{-1} nonzero and we need to discuss their limits as these cutoffs are removed.

The main points of this analysis are the following. Acting on translation-invariant functions, $\mathcal{M}_N$ becomes

$$\mathcal{M}_N = -\nu\sum_{n=1}^{N}\Delta_{\mathbf{x}_n} + \sum_{n<n'}\tilde{\mathcal{D}}^{ij}(\mathbf{x}_n - \mathbf{x}_{n'})\partial_{x_n^i}\partial_{x_{n'}^j} \qquad (18)$$

possessing an $m \to 0$ and $\nu \to 0$ limit which is a singular elliptic operator $\mathcal{M}_N^{\mathrm{sc}}$. It can be shown [10] that the Green functions occurring in (17) have limits which are well defined in the UV and which render (17) finite for L finite (and similarly for higher point functions). Thus we need to find the leading behavior of (17) as $L \to \infty$ with $\mathcal{M}_{2N}$ replaced by $\mathcal{M}_{2N}^{\rm sc}$.

Recalling that $F_2 = \mathcal{M}_2^{-1} C_L$, it is convenient to view Eq. (17) as a differential equation for F_4 that becomes for the connected part $F_4^c \equiv F_4 - F_2 \otimes F_2$,

$$\mathcal{M}_4 F_4^c = \mathcal{L}(F_2 \otimes F_2), \tag{19}$$

where $-\mathcal{L}$ is given by the sum in (18) with $n = 1, 2$ and $n' = 3, 4$. By (15), the RHS of (19) has a well defined limit as $L \to \infty$ given by

$$\frac{\epsilon^2}{9(2-\kappa)^2 D_1^2} \mathcal{L}\, |\mathbf{X}|^{2-\kappa} |\mathbf{Z}|^{2-\kappa} \tag{20}$$

and is a homogeneous (rotationally invariant) function of $\mathbf{X} \equiv \mathbf{x}_1 - \mathbf{x}_2, \mathbf{Y} \equiv \mathbf{x}_2 - \mathbf{x}_3$, and $\mathbf{Z} \equiv \mathbf{x}_3 - \mathbf{x}_4$ of degree $2 - \kappa$. One easily checks that

$$F_4^{c,\rm sc} = \frac{\epsilon^2}{6(2-\kappa)^2(5-\kappa) D_1^2} \left(|\mathbf{X}|^{2(2-\kappa)} + |\mathbf{Z}|^{2(2-\kappa)}\right) - \frac{\epsilon^2}{9(2-\kappa)^2 D_1^2} |\mathbf{X}|^{2-\kappa} |\mathbf{Z}|^{2-\kappa} \tag{21}$$

solves the limiting case of Eq. (19). We deduce that $\mathcal{M}_4^{\rm sc}(F_4^c - F_4^{c,\rm sc}) \to 0$ as $L \to \infty$. By scale invariance, it is thus reasonable to conclude that the solution for finite but large L should differ from the universal scaling form by zero modes of $\mathcal{M}_4^{\rm sc}$ so that

$$F_4^c - \sum_{0 \le \rho_{4,n} \le 2(2-\kappa)} L^{\rho_{4,n}} \sum_m \gamma_{nm} F_{4,nm}^c \underset{L\to\infty}{\longrightarrow} F_4^{c,\rm sc}, \tag{22}$$

where $F_{4,nm}^c$ are homogeneous zero modes of $\mathcal{M}_4^{\rm sc}$ of degree $2(2-\kappa) - \rho_{4,n}$ and the nonuniversal coefficients γ_{nm} depend on the source covariance C.

In fact, using spectral analysis of $\mathcal{M}_4$ [10], (22) can be made rigorous. Similar analysis can be repeated for N-point correlators: Nonuniversal L-dependent terms proportional to homogeneous zero modes of $\mathcal{M}_N^{\rm sc}$ can be present in the large L asymptotics. We thus face the problem of finding such zero modes, of determining whether they are present in the N-point function of T, and finally of finding whether they contribute to the structure function S_N. We will now show that at least for small κ the zero modes are present and dominate the structure functions.

At $\kappa \to 0$, $\tilde{D}^{ij} = 2D_1 \delta^{ij}$ [having finite D_1 requires the vanishing of D_0 as $\kappa \to 0$ in order to renormalize the ultraviolet divergence in (4); D_0 will never show up below]. We immediately obtain for the $\kappa = 0$ operators

$$\mathcal{M}_{2,0}^{\rm sc} = 2D_1 \nabla_{\mathbf{x}_1} \cdot \nabla_{\mathbf{x}_2} = -2D_1 \Delta_{\mathbf{X}}, \tag{23}$$

$$\begin{aligned} \mathcal{M}_{4,0}^{\rm sc} &= 2D_1 \sum_{1 \le n < n' \le 4} \nabla_{\mathbf{x}_n} \cdot \nabla_{\mathbf{x}_{n'}} \\ &= -D_1 \sum_{n=1}^{4} \Delta_{\mathbf{x}_n} \\ &= -2D_1(\Delta_{\mathbf{X}} + \Delta_{\mathbf{Y}} + \Delta_{\mathbf{Z}} - \nabla_{\mathbf{X}} \cdot \nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}} \cdot \nabla_{\mathbf{Z}}) \end{aligned} \tag{24}$$

in the difference variables $\mathbf{X}, \mathbf{Y}$, and $\mathbf{Z}$ and using the subscript to refer to $\kappa = 0$. In fact, the higher point correlators reduce at $\kappa = 0$ to the standard Gaussian expression with sums of products of the two-point functions.

We shall find the homogeneous zero modes of the operator $\mathcal{M}_4^{\rm sc}$ in perturbation expansion in powers of κ. Equation (5) implies that (for $m = 0$) $\tilde{D}^{ij}(\mathbf{x}) = 2D_1[\delta^{ij} + \kappa R^{ij}(\mathbf{x})] + \mathcal{O}(\kappa^2)$ with

$$R^{ij}(\mathbf{x}) = \delta^{ij}(\tfrac{1}{2} + \ln|\mathbf{x}|) - \tfrac{1}{2} x^i x^j |\mathbf{x}|^{-2}. \tag{25}$$

Hence, to the first order in κ, $\mathcal{M}_4^{\rm sc} = \mathcal{M}_{4,0}^{\rm sc} + 2\kappa D_1 V_4$, with

$$\begin{aligned} V_4 = &- R^{ij}(\mathbf{X})\partial_{X^i}\partial_{X^j} - R^{ij}(\mathbf{Y})\partial_{Y^i}\partial_{Y^j} - R^{ij}(\mathbf{Z})\partial_{Z^i}\partial_{Z^j} \\ &- [R^{ij}(\mathbf{X}+\mathbf{Y}) - R^{ij}(\mathbf{X}) - R^{ij}(\mathbf{Y})]\partial_{X^i}\partial_{Y^j} \\ &- [R^{ij}(\mathbf{Y}+\mathbf{Z}) - R^{ij}(\mathbf{Y}) - R^{ij}(\mathbf{Z})]\partial_{Y^i}\partial_{Z^j} \\ &- [R^{ij}(\mathbf{X}+\mathbf{Y}+\mathbf{Z}) - R^{ij}(\mathbf{X}+\mathbf{Y}) \\ &- R^{ij}(\mathbf{Y}+\mathbf{Z}) + R^{ij}(\mathbf{Y})]\partial_{X^i}\partial_{Z^j}. \end{aligned} \tag{26}$$

We shall search for the homogeneous zero modes of $\mathcal{M}_4^{\rm sc}$ symmetric under three-dimensional translations and rotations and under permutations of four points, which is a standard and mathematically sound perturbative problem for operators with discrete spectrum. The symmetric zero modes of the lowest homogeneity of $\mathcal{M}_{4,0}^{\rm sc}$ occur in degree zero (constants) and in degree four. The latter has the form

$$a \sum_{\{n,n'\}} (\mathbf{x}_n - \mathbf{x}_{n'})^4 + b \sum_{\{\{n,m\},\{n,m'\}\}} (\mathbf{x}_n - \mathbf{x}_m)^2 (\mathbf{x}_n - \mathbf{x}_{m'}) + c \sum_{\{\{n,n'\},\{m,m'\}\}} (\mathbf{x}_n - \mathbf{x}_{n'})^2 (\mathbf{x}_m - \mathbf{x}_{m'})^2 \equiv aF_1 + bF_2 + cF_3, \tag{27}$$

where the pairs $\{n, n'\}$ and $\{m, m'\}$ are assumed different, as well as the pairs $\{n, m\}$ and $\{n, m'\}$, and where $10a + 14b + 3c = 0$.

The constant survives as the eigenvalue of $\mathcal{M}_4^{\rm sc}$ for $\kappa \neq 0$. Thus we need to calculate in degenerate perturbation theory how the fourth degree zero modes change with κ. For this we write $\mathcal{M}_{4,0}^{\rm sc} = -2D_1(\Delta_{\tilde{\mathbf{X}}} + \Delta_{\tilde{\mathbf{Y}}} + \Delta_{\tilde{\mathbf{Z}}})$, where $\tilde{\mathbf{X}} = \mathbf{X}, \tilde{\mathbf{Y}} = \sqrt{2}(\mathbf{Y} + \frac{1}{2}\mathbf{X} + \frac{1}{2}\mathbf{Z})$, and $\tilde{\mathbf{Z}} = \mathbf{Z}$. Denoting $R \equiv (\tilde{\mathbf{X}}^2 + \tilde{\mathbf{Y}}^2 + \tilde{\mathbf{Z}}^2)^{1/2} = [\frac{1}{2}\sum_{\{n,n'\}}(\mathbf{x}_n - \mathbf{x}_{n'})^2]^{1/2}$, we obtain

$$\mathcal{M}_{4,0}^{\rm sc} = -\frac{2D_1}{R^8}\partial_R R^8 \partial_R - \frac{2D_1}{R^2}\Phi, \tag{28}$$

where Φ is the Laplacian on the sphere S^8 in the space of $(\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}})$. Now pick two linearly independent zero modes $R^4 f_i$, $i = 1,2$, of the form (27) and look for a homogeneous zero mode

$$R^{4+\lambda\kappa+\mathcal{O}(\kappa^2)}\{[a_1 + \mathcal{O}(\kappa)]f_1 + [a_2 + \mathcal{O}(\kappa)]f_2 + \kappa f_3 + \mathcal{O}(\kappa^2)\} \quad (29)$$

with a homogeneous degree zero function f_3 orthogonal to $f_{1,2}$ in $L^2(S^8)$. We obtain in the linear order in κ

$$\mathcal{M}_{4,0}^{\mathrm{sc}}[\lambda R^4 \ln R(a_1 f_1 + a_2 f_2) + R^4 f_3] + 2D_1 V_4 R^4(a_1 f_1 + a_2 f_2) = 0, \quad (30)$$

or, using the form (28) of $\mathcal{M}_{4,0}^{\mathrm{sc}}$,

$$-15\lambda(a_1 f_1 + a_2 f_2) - 44 f_3 - \Phi f_3 + \frac{1}{R^2} V_4 R^4(a_1 f_1 + a_2 f_2) = 0. \quad (31)$$

Upon taking the $L^2(S^8)$ scalar products with $f_{1,2}$, f_3 drops out resulting in the relation

$$-15\lambda \sum_{j=1,2} (f_i, f_j) a_j + \sum_{j=1,2'} \left(f_i, \frac{1}{R^2} V_4 R^4 f_j\right) a_j = 0. \quad (32)$$

For the explicit calculation we took $R^4 f_1 = 3F_1 - 10F_3$ and $R^4 f_2 = -7F_1 + 5F_2$. The integrals over the eight-dimensional sphere are most conveniently done by using homogeneity to transform them to Gaussian integrals over $\mathbf{R}^9$. The integration is straightforward with MAPLE and the matrix in (32) becomes proportional to

$$\begin{pmatrix} -52 - 25\lambda & 15 + 15\lambda \\ 18 + 15\lambda & -20 - 20\lambda \end{pmatrix}$$

with the eigenvalues $\lambda_1 = -14/5$ and $\lambda_2 = -1$. The corresponding eigenfunctions (29) are $F_{4,1}^c = R^{-(2+4/5)\kappa+\mathcal{O}(\kappa^2)}[F_1 - 2F_2 + 6F_3 + \mathcal{O}(\kappa)]$ and $F_{4,2}^c = R^{-\kappa+\mathcal{O}(\kappa^2)}[-7F_1 + 5F_2 + \mathcal{O}(\kappa)]$. The $\mathcal{O}(\kappa)$ contributions κf_3 may be calculated from Eq. (31) and are continuous functions on S^8 (no logarithmic divergences). The latter holds to all orders and also nonperturbatively for small κ allowing conclusions about the four-point functions with coinciding points like $S_4(\mathbf{x})$.

For large L the connected four-point function takes the form

$$\left\langle \prod_{n=1}^{4} T(\mathbf{x}_n) \right\rangle^c = \gamma_{4,0} L^{4-2\kappa} + \gamma_{4,1} L^{4\kappa/5+\mathcal{O}(\kappa^2)} F_{4,1}^c + \gamma_{4,2} L^{-\kappa+\mathcal{O}(\kappa^2)} F_{4,2}^c + \left\langle \prod_{n=1}^{4} T(\mathbf{x}_n) \right\rangle^{c,\mathrm{sc}} + \mathcal{O}[(L/R)^{-2+\mathcal{O}(\kappa)}] \quad (33)$$

uniformly in small κ. Since the connected correlation vanishes and $\langle \prod_{n=1}^4 T(\mathbf{x}_n)\rangle^{c,\mathrm{sc}}$ reduces to $\frac{\epsilon^2}{360 D_1^2}(3F_1 - 10F_3)$ for $\kappa = 0$, we infer that $\gamma_{4,0} = \mathcal{O}(\kappa)$, $\gamma_{4,1} = \frac{\epsilon^2}{216 D_1^2} + \mathcal{O}(\kappa)$, and $\gamma_{4,2} = \frac{\epsilon^2}{540 D_1^2} + \mathcal{O}(\kappa)$. The result (6) for $N = 2$ follows with $\gamma_4 = \frac{\epsilon^2}{3D_1^2} + \mathcal{O}(\kappa)$ and ρ_4 given by (7) since only the $F_{4,1}^c$ term gives nonzero contribution to the structure function. The zero mode $F_{4,2}^c$ corresponding to $\lambda = -1$ is actually obtained from a zero mode of $\mathcal{M}_3$ by extending it to a function of four $\mathbf{x}_i$'s by symmetrizing. This is a general feature: Zero modes of $\mathcal{M}_{N'}$ give rise to zero modes of $\mathcal{M}_{2N}$ for $2N > N'$. These, however, do not contribute to the structure functions S_{2N}. The only zero mode of $\mathcal{M}_{2N}$ that contributes is the unique one not coming from the lower dimensional $\mathcal{M}_{N'}$'s, the one that at $\kappa = 0$ is obtained from the monomial $\prod_{i=1}^N (\mathbf{x}_{2i-1} - \mathbf{x}_{2i})^2$ by symmetrizing and subtracting partial traces. It gives rise to the dominant contribution to S_{2N} which has to be present by the Hölder inequality. In particular, all exponents ρ_{2N} should be $\mathcal{O}(\kappa)$ for small κ.

The asymptotic behavior of the scalar correlation functions encodes subtle information about the behavior of the Green functions of the singular multibody operators $\mathcal{M}_N$ with continuous spectrum. The reduction of its study to that of discrete spectrum operators given by $\mathcal{M}_N$'s acting on homogeneous functions should be thought of as realizing a renormalization group type approach to the model, with the homogeneous zero modes of $\mathcal{M}_N$ playing the role of relevant interactions. This may be the most important hint from the above exact solution for the anomalous scaling of the passive scalar.

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