Semiclassical Description of Tunneling in Mixed Systems: Case of the Annular Billiard

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We study quantum-mechanical tunneling between symmetry-related pairs of regular phase space regions that are separated by a chaotic layer. We consider the annular billiard, and use scattering theory to relate the splitting of quasidegenerate states quantized on the two regular regions to specific paths connecting them. The tunneling amplitudes involved are given a semiclassical interpretation by extending the billiard boundaries to complex space and generalizing specular reflection to complex rays. We give analytical expressions for the splittings, and show that the dominant contributions come from *chaos-assisted* paths that tunnel into and out of the chaotic layer.

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Recently there has been a surge of interest in quantum systems whose classical counterparts exhibit a mixture of regular and chaotic motion. Such systems are of great importance since they comprise the majority of dynamical systems found in nature. One interesting feature is quantum-mechanical tunneling between regular regions in phase space that are separated by a chaotic layer. Numerical studies of mixed systems have shown that the splitting of quasidoublets associated with such pairs of regular regions was much higher than could be explained by direct tunneling processes [1–3]. This was attributed in [2,3] to a suggested mechanism of *chaos-assisted tunneling*, i.e., tunneling from one regular region into the chaotic sea, propagating "classically" to the other side, and tunneling out into the second regular region. Since a large part of the phase space is thus traversed via classically allowed transitions, these paths were expected to considerably enhance the splitting.

In this work we study tunneling from a scattering theory point of view, taking as a specific example the annular billiard proposed in [3]. This allows us to write the splittings as a sum over paths in angular momentum space, which can include both classically allowed and tunneling transitions. The tunneling amplitudes are evaluated semiclassically by continuing the billiard boundaries to complex configuration space, and generalizing the mechanism of specular reflection to the case of complex rays. We obtain analytical expressions for the splittings in terms of these transition amplitudes, and show that the dominant contributions arise from paths that tunnel into (and out of) the chaotic layer via intermediate angular momenta, which lie on the boundary between the regular and chaotic regions. This approach yields values for the splitting that are in good agreement with exact results.

The annular billiard consists of the space between two nonconcentric circles of radius R and $a < R$, centered at (x, y) coordinates $O \equiv (0,0)$ and $O' \equiv (\delta, 0)$, respectively. Classically, a particle moves freely between specular reflections on the bounding circles. We parametrize trajectories by their impact parameter L with respect to O and their direction of propagation γ. Trajectories of $|L| > a + \delta$ do not hit the inner circle, but rotate forever at constant L, while the phase space for $|L| < a + \delta$ consists of a mixture of regular islands and chaotic layers. Phase space plots can be found in Ref. [3]. In this Letter we use parameters for which a single chaotic layer extends from $L = a + \delta$ to $L = -(a + \delta)$.

We apply the scattering approach to quantization first proposed in [4]. Let us write the wave function in terms of incoming and outgoing cylindrical waves,

$$\psi(r, \phi) = \sum_{n=-\infty}^{\infty} [\alpha_n H_n^{(2)}(kr) + \beta_n H_n^{(1)}(kr)] e^{in\phi},$$

where $H_n^{(1,2)}(x)$ denotes the Hankel functions of the first and second kinds, and k is the wave number. We consider the billiard as two back-to-back scattering systems; the first, "inner" system consists of incoming waves being reflected to outgoing waves by the exterior of the inner circle, whereas the "outer" system consists of outgoing waves being scattered to incoming waves by the interior of the outer circle. The two systems are characterized by scattering matrices $S^{(I,O)}(k)$, respectively, that relate the coefficient vectors α, β by $\boldsymbol{\beta} = S^{(I)}(k)\boldsymbol{\alpha}$ and $\boldsymbol{\alpha} = S^{(O)}(k)\boldsymbol{\beta}$. Requiring the two relations to be consistent results in the quantization condition

$$\det[S(k) - 1] = 0, \qquad S(k) = S^{(I)}(k) S^{(O)}(k). \quad (1)$$

Thus the billiard supports an eigenvalue whenever one of the eigenphases of $S(k)$ equals an integer multiple of 2π.

We drop the wave number argument in the sequel. $S^{(O)}$ is clearly just the diagonal matrix $S_{nm}^{(O)} = -H_n^{(1)}(kR)/H_n^{(2)}(kR)\delta_{nm}$, while $S^{(I)}$ can be obtained by writing it as $S_{nm}^{\prime(I)} = -H_n^{(2)}(ka)/H_n^{(1)}(ka)\,\delta_{nm}$ in the primed coordinates, and then performing a coordinate change. Using the addition theorem for Bessel functions (see, e.g., [5]) this gives

$$S_{nm}^{(I)} = -\sum_{\ell=-\infty}^{\infty} J_{n-\ell}(k\delta) J_{m-\ell}(k\delta) \frac{H_\ell^{(2)}(ka)}{H_\ell^{(1)}(ka)}. \quad (2)$$

The structure of S can be seen in Fig. 1, which shows $|S_{nm}|$ for $k = 100$, $a = 0.4$, and $\delta = 0.2$. The inset gives an overview of one quadrant. One can see that S is almost diagonal at angular momenta $n, m > k(a + \delta)$. For n, m below this value most of the amplitude lies within the region of classically allowed transitions and is delimited by caustics due to classical rainbow scattering. Closer inspection reveals that these ridges extend into the classically forbidden region as well, although they are exponentially suppressed there. The main figure shows a single row $n = 70$. Since $n > k(a + \delta)$, the diagonal element is almost unimodular, while the nondiagonal elements are exponentially small. One finds a maximum at $m \approx 63$ that corresponds to the ridge seen in the inset, while away from this maximum the matrix elements decay faster than exponentially.

For our purpose it will prove sufficient to approximate the magnitudes $|S_{nm}| = |S_{nm}^{(I)}|$. We will now sketch this semiclassical derivation; a more detailed account will be given elsewhere [6]. Starting from (2), we write $J_{m-\ell}(k\delta) = e^{i\pi(m-\ell)} J_{\ell-m}(k\delta)$ and apply the Poisson summation formula to get

$$S_{nm}^{(I)} = -\sum_{\mu=-\infty}^{\infty} \int_{-\infty}^{\infty} d\ell\, e^{i2\pi\mu\ell + i\pi(m-\ell)} \times J_{n-\ell}(k\delta) J_{\ell-m}(k\delta) \frac{H_\ell^{(2)}(ka)}{H_\ell^{(1)}(ka)}, \qquad (3)$$

and consider only the $\mu = 0$ term (see below). We replace the Bessel functions by their Sommerfeld representation

$$J_\nu(z) = (1/2\pi) \int_C d\gamma\, \exp[iz\cos\gamma + i\nu(\gamma - \pi/2)],$$

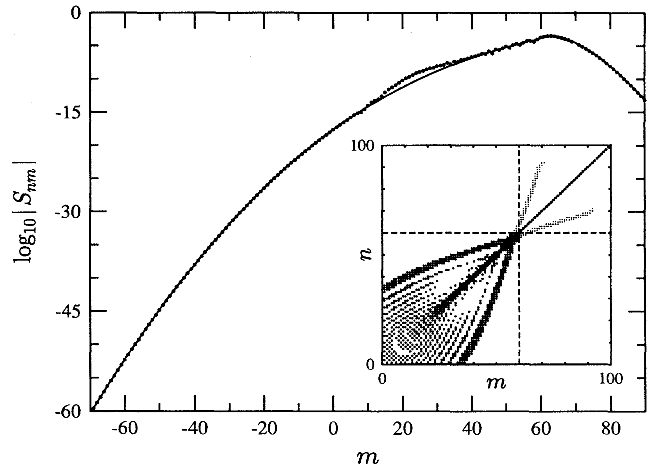

FIG. 1. Tunneling amplitudes $|S_{nm}|$ for $n = 70$, $a = 0.4$, $\delta = 0.2$, and $k = 100$ as a function of m, calculated exactly (dots) and semiclassically (full line). The inset shows one quadrant of $|S|$, with larger values corresponding to larger dots, in arbitrary units. Off-diagonal ridges at $n, m > k(a + \delta)$ are exponentially enhanced in the plot. The dashed lines indicate $k(a + \delta)$.

where the contour of integration C runs from $-\pi + i\infty$ to $\pi + i\infty$. The contribution of the Hankel functions to nondiagonal elements of $S^{(I)}$ is asymptotically given by their Debye approximation [7] to be $i\exp[i\,2ka(\sin Q - Q\cos Q)]$, where $Q = \arccos(\ell/ka)$. $S_{nm}^{(I)}$ can then be written in the form of an integral over an exponential whose argument is proportional to k. In the semiclassical limit $k \gg 1$ this integration can be carried out using the saddle point approximation, giving

$$S_{nm}^{(I)} \approx \sum_p \sqrt{\frac{|\mathcal{R}_p|}{k}}\, e^{ik\Phi_p + (i/2)\arg\mathcal{R}_p - i3\pi/4},$$

$$\Phi_p = \delta(\cos\gamma_{f,p} - \cos\gamma_{i,p}) - \frac{m}{k}\left(\gamma_{f,p} + \frac{\pi}{2}\right) + \frac{n}{k}\left(\gamma_{i,p} + \frac{\pi}{2}\right) + 2a\sin\left(\frac{\gamma_{f,p} - \gamma_{i,p}}{2}\right),$$

$$\mathcal{R}_p = \frac{1}{2\pi}\, \frac{\partial^2\Phi}{\partial\gamma_f\partial\gamma_i} \left[\frac{\partial^2\Phi}{\partial\gamma_f^2}\frac{\partial^2\Phi}{\partial\gamma_i^2} - \left|\frac{\partial^2\Phi}{\partial\gamma_f\partial\gamma_i}\right|^2\right]_p^{-1}, \qquad (4)$$

where the summation is over saddles p in the complex Q, γ_i, γ_f space. These saddles are determined by

$$0 = \gamma_{f,p} - \gamma_{i,p} + 2Q_p, \qquad (5a)$$

$$m = n - k\delta(\sin\gamma_{f,p} - \sin\gamma_{i,p}), \qquad (5b)$$

$$n = -k\delta\sin\gamma_{i,p} + ka\cos Q_p. \qquad (5c)$$

Solving (5) yields a number of saddle points in the complex plane. For most values of n, m the dominant contribution comes from a single saddle, whereas others are either negligible or cannot be reached by deformation of the contour of integration. This dominant saddle coalesces with its symmetry-related counterpart at values of n, m, which lie close to the ridges seen in the inset of Fig. 1. (The actual ridges correspond to points where $\gamma_{f,p}$ becomes real and equal to $\pi/2$ or $3\pi/2$.) In principle, near this caustic saddle point integration should be replaced by a uniform approximation, and also the Hankel functions should be treated more carefully. However, a detailed treatment of this region is not required for the purpose of this paper. The full line in Fig. 1 shows the contribution of the dominant saddle to $|S_{nm}^{(I)}|$, and one can see that there is excellent agreement over a wide range of angular momenta.

For completeness, let us consider the $\mu \neq 0$ terms in (3). Each integral in the summation can be written as a residue sum over the poles of $H_\ell^{(2)}(ka)/H_\ell^{(1)}(ka)$.

These residue contributions are interpreted in standard diffraction theory as surface waves excited on the inner disk by rays of grazing incidence [8]. As can be seen from Fig. 1, at our parameter values they give rise to significant corrections only over a small range of m. As the corresponding transitions will turn out to be of no interest in the present context, we defer a discussion of the residue terms to [6].

The approximation given in Eqs. (4) and (5) can be interpreted in terms of classical trajectories in complex configuration space, by extending the dynamics inside the billiard to complex coordinates. The extension of free flight to complex coordinates is trivial. In order to describe the interaction with the inner circle, we define a complex circle of radius a around the point (x_0, y_0) as the surface for which $x - x_0 = a\cos\beta$, $y - y_0 = a\sin\beta$, where β is allowed to be complex. Our motivation is that a wave function satisfying Dirichlet or Neumann boundary conditions on such a circle for real β will also satisfy them for complex β. Next, we note that through (almost) every point on a circle one can pass two distinct trajectories with a given impact parameter relative to its center. Since reflection from a circle centered at the origin preserves the impact parameter, we define specular reflection as the mapping from one of these trajectories to the other.

It is easy to verify that any given initial and final impact parameters are linked by at least one complex trajectory. The conditions specifying the corresponding initial and final angles are then equivalent to Eqs. (5). Thus $\gamma_{i,p}$ and $\gamma_{f,p}$ have the meaning of the initial and final angles for which an incoming trajectory with impact parameter $L_i = n/k$ is reflected off the complex inner circle to the impact parameter $L_f = m/k$. Moreover, (5a) is the generalized specular reflection condition, while (5b) and (5c) give the classical deflection function. Note that the impact parameter with respect to the center of the inner circle is given by $L' = a\cos Q$. By (5c), we see that L' can become complex even if L_i and L_f are real. Furthermore, unlike in [9], both the initial *and* final angles are generically complex.

Finally, it is straightforward to show that the phase Φ in (4) is given by the reduced action (*not* the length) of the ray, $\Phi = -\int_{t_i}^{t_f} dt\,[r(t)\dot{p}_r(t) + \varphi(t)\dot{L}(t)]$, where (r,φ), (p_r, L) are canonically conjugate polar coordinates and momenta, and that $|\mathcal{R}_p|$ is the corresponding reciprocal stability $|\partial^2\Phi/\partial L_i \partial L_f|/2\pi$. We therefore find that (4) coincides with the sum over classical trajectories, which constitutes the usual semiclassical approximation to scattering matrix elements (see [10]).

We will now extract the level splitting. We first examine the splittings in eigenphases, and will regard energy splittings later. Let $|\pm\rangle$ represent the doublet of eigenvectors peaked at angular momenta $\pm n$,

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|\,n\rangle \pm |-n\rangle) + \sum_m \kappa_m^{\pm}\,|\,m\rangle,$$

where the $\kappa_m^{\pm}$ are exponentially small. We denote the corresponding eigenphases by $\theta_n^{\pm}$ and their splitting by $\delta\theta_n = |\theta_n^+ - \theta_n^-|$. Using $\exp(iN\theta_n^{\pm}) = \langle \pm|S^N|\pm\rangle$ we get

$$\sin\left(\frac{N}{2}\,\delta\theta_n\right) = |[S^N]_{-n,n}| + C_n^{(N)},$$

which describes the tunneling oscillations between n and $-n$, with a correction term $C_n^{(N)} \lesssim k^2 \max|\kappa_m^{\pm}|^2$. For $C_n^{(N)} \ll N\,\delta\theta_n \ll 1$, we can therefore write

$$\delta\theta_n \approx \frac{2}{N}\,|[S^N]_{-n,n}| = \frac{2}{N}\left|\sum_{\{\lambda_i\}}\prod_{i=1}^{N-1} S_{\lambda_i,\lambda_{i+1}}\right|. \quad (6)$$

Equation (6) relates the splitting to a sum over all paths $\{\lambda_i\}_{i=1}^N$ in matrix element space that go from $\lambda_1 = -n$ to $\lambda_N = n$. The simplest such paths involve only the direct tunneling amplitude $S_{-n,n}$, which from Fig. 1 is exceedingly small. A much larger contribution comes from paths that tunnel only over small distances in angular momentum space and traverse the remaining distance via classically allowed transitions. These are the chaos-assisted tunneling paths proposed in [2]. In order to deal with such contributions, we first need to consider the structure of the chaotic part of phase space.

The quantum evolution of systems with purely chaotic classical counterparts is usually very well described by random matrix theory [11]. However, this is only appropriate when phase space can be assumed to be completely structureless. In the case of a classically mixed system, this is an oversimplification: As the Lyapunov exponent vanishes smoothly at the interface between chaotic and regular phase space regions, there is always an intermediate layer in which chaotic, but relatively stable, motion gives rise to classical staying times much longer than the mean level density. Moreover, angular momentum remains a preferred basis well into the chaotic part of phase space. This corresponds to the observation made in [2,12] that classical regular structure can be quantum mechanically continued into the chaotic sea. Consequently, it is only the internal part $|l| \leq l_{\mathrm{COE}}$ of the chaotic sea that is appropriately modeled by a random matrix ensemble, which in the present case is a circular orthogonal ensemble (COE) of size $\sim 2l_{\mathrm{COE}}$.

It turns out that there are two types of chaos-assisted paths that contribute dominantly to the splitting: (I) paths $(n,\gamma,-n)$ that tunnel from n directly into the chaotic region, propagate in some eigenstate $|\gamma\rangle$ of the internal block, and finally tunnel to $-n$, and (II) paths $(n,l,\gamma,-l,-n)$ that propagate from n to $|\,\gamma\,\rangle$ and then to $-n$ in two jumps via intermediate edge angular momenta l and $-l$, respectively. We will first consider paths of type (I). Let us denote the eigenphase of $|\,\gamma\,\rangle$ by θ_γ, and let $\theta_n = -i\ln S_{n,n}$. Summing over the possible dwell times at n and γ gives contributions of the type $S_{n,\gamma}S_{\gamma,-n}/\sin[(\theta_\gamma - \theta_n)/2]$ to the splitting. Assuming that the $S_{n,\gamma}$ are distributed independently of the θ_γ, we

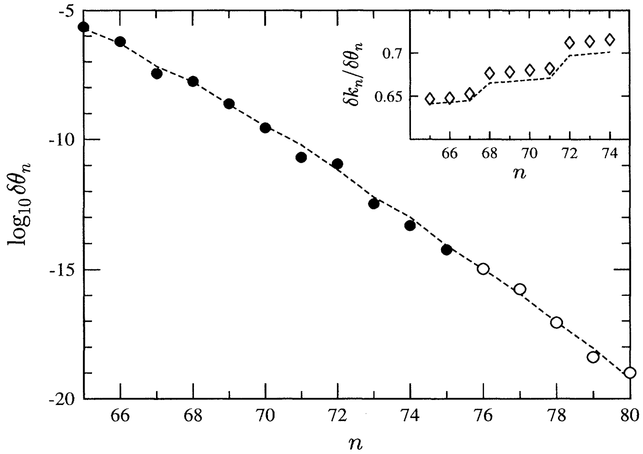

FIG. 2. Median eigenphase splittings $\delta\theta_n$ obtained by diagonalization of S (full circles), using Eq. (6) with $N = 2^{13}$ (empty circles), and as estimated from Eqs. (7) and (8) (dashed line) with $c = 0.1$ (see text). Inset: $\delta k_n/\delta\theta_n$ (diamonds) and $|\partial\theta_n^{(0)}/\partial k|^{-1}$ (dashed line) evaluated around $k = 100$.

can average over the COE and calculate the median of the type (I) splitting contributions [13]

$$\delta\theta_n^{(I)} = \left| \sum_{\gamma} \frac{S_{n,\gamma}\, S_{\gamma,-n}}{\sin[(\theta_\gamma - \theta_n)/2]} \right| \sim \frac{8}{\pi}\, v_n^2, \qquad (7)$$

where $v_n^2 = \langle \sum_\gamma |S_{n,\gamma}|^2 \rangle$, which we approximate by $v_n^2 \approx \sum_{g=-l_{\mathrm{COE}}}^{l_{\mathrm{COE}}} |S_{n,g}|^2$. Similarly, the type (II) contributions can be estimated by

$$\delta\theta_n^{(\mathrm{II})} = \frac{1}{4} \left| \sum_{l,\gamma} \frac{S_{n,l}\, S_{-l,-n}}{\sin^2[(\theta_l - \theta_n)/2]} \, \frac{S_{l,\gamma}\, S_{\gamma,-l}}{\sin[(\theta_\gamma - \theta_n)/2]} \right|$$

$$\sim \frac{2}{\pi} \left(\sum_l \left| \frac{|S_{n,l}|^2\, v_l^2}{\sin^2[(\theta_l - \theta_n)/2]} \right|^2 \right)^{1/2}. \qquad (8)$$

Note that $\delta\theta_n$ will fluctuate strongly about the median values due to avoided crossings [2,3,14]. The relative importance of the contributions $\delta\theta_n^{(\mathrm{I,II})}$ is system specific; for the present system we find $\delta\theta_n^{(\mathrm{II})}/\delta\theta_n^{(\mathrm{I})} \sim 25$. Furthermore, paths that include additional transitions outside the chaotic block give negligible contributions.

It remains to connect eigenphase and eigenvalue splittings. From (7) and (8) we see that $\delta\theta_n(k)$ is approximately constant over exponentially small ranges of k. It is then simple to show that $\delta k_n \approx |\partial\theta_n^{(0)}/\partial k|^{-1}\delta\theta_n$, where $\theta_n^{(0)}$ is evaluated at $\delta = 0$. However, we expect Eqs. (7) and (8) to overestimate the actual splittings: our model neglects residual transport barriers inside the chaotic block, and the damping effect of imaginary parts of eigenphases is not accounted for. As these errors are expected to be independent of n, we can correct by an overall factor c that we extract from the numerical data.

In Fig. 2 we present a numerical test of our results. We plot median values of $\delta\theta_n$ that were obtained by varying the outer radius R over 30 values between 1 and 1.3. Note that changing R leaves transition probabilities constant. The full circles represent the exact $\delta\theta_n$, as a function of n. Splittings smaller than $\sim 10^{-14}$ (denoted by empty circles) could not be calculated directly due to the finite precision of the diagonalization procedure, but were obtained by calculating S^N for large N and applying Eq. (6). Finally, the dashed line shows the prediction of Eqs. (7) and (8) for $l_{\mathrm{COE}} = 50$ and $c = 0.1$. While the n dependence due to tunneling is reproduced very well, the coefficient due to transport across the chaotic sea could only be calculated up to an order of magnitude. The inset shows $\delta k_n/\delta\theta_n$ evaluated from the quantization condition (1) around $k = 100$ for $R = 1$, compared to $|\partial\theta_n^{(0)}/\partial k|^{-1}$. We see that the correspondence is good.

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Note added.—After submission of this Letter we received a preprint [15] that deals with related topics.

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