Multiparticle Dynamics in an Expanding Universe

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Approximate equations of motion for multiparticle systems in an expanding Einstein-deSitter universe are derived from the Einstein-Maxwell field equations using the Einstein-Infeld-Hoffmann surface integral method. At the Newtonian level of approximation one finds that, in comoving coordinates, both the Newtonian gravitational and Coulomb interactions in these equations are multiplied by the inverse third power of the scale factor $R(t)$ appearing in the Einstein-deSitter field and they acquire a cosmic "drag" term. Nevertheless, both the period and luminosity size of bound two-body systems whose period is small compared to the Hubble time are found to be independent of t.

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It is usually assumed that local systems such as atoms, the solar system, and even galaxies are unaffected by the cosmic expansion of our universe since, it is argued, if everything expanded equally, the expansion would be unobservable. Misner, Thorne, and Wheeler [1] (MTW) have likened the situation to pennies attached to the surface of a balloon. As the balloon expands the distance between the pennies increases while their size remains fixed. This being the case it follows that there must be some maximum scale, assumed by MTW to be the distance between clusters of galaxies, below which systems are not affected by the expansion. As reasonable as this view appears, it nevertheless raises at least as many questions as it purports to answer. What determines this maximum scale? What "shields" systems smaller than it from the cosmic expansion? Do these systems live in their own space-time and if so how do they communicate with the cosmic space-time? And what if we were to paint dots on MTW's balloon instead of affixing pennies to it? I am going to argue that there is only one space-time and that all physical systems, big or small, feel the effect of the cosmic expansion in one way or another. The problem then is to determine the effect of this expansion.

If one knew the dynamical equations of motion of various types of multiparticle systems in an expanding universe then of course one could determine what effect the expansion has on their evolution. But simply to postulate some such equations begs the question. There is, however, a procedure for deriving these equations directly from the Einstein-Maxwell field equations of general relativity without the need of any *ad hoc* assumptions. This procedure is the Einstein-Infeld-Hoffmann [2] (EIH) surface integral method. While not well known, it is in fact the only way to be sure that the equations of motion used for a given system are compatible with the field equations of general relativity. To obtain these equations one integrates the field equations (Einstein for uncharged particles, Einstein-Maxwell for charged particles) over small surfaces surrounding each particle in the system. The requirement that each integral be independent of the particular surface chosen leads to the equations of motion. As such the EIH method does not use singular right-hand sides of the field equations as do other methods nor does it encounter infinities that must be renormalized.

To evaluate the integrands on the surfaces one of course needs to know the fields on these surfaces. Since the field equations are nonlinear it is not possible to obtain exact solutions for a given distribution of sources. The best one can do in general is to obtain approximate solutions. The resulting equations of motion are therefore also only approximate and only apply to systems for which the approximations are valid. In their original work, Einstein and his collaborators perturbed the gravitational field about its "flat" Minkowski values and assumed that their systems were "slow," that is, the light travel time T_L across the system is small compared to characteristic times T_S of the system such as its period. In the case of an expanding universe one perturbs one of the Robertson-Walker-Lemaître fields; in this Letter I give the results for perturbing the Einstein-deSitter field given by

$$g_{\mu\nu} = \mathrm{diag}[1, -R^2(t), -R^2(t), -R^2(t)]$$

where $R(t) = (t/t_0)^{2/3}$ and t is the cosmic time. In what follows I will assume that T_S is less than the Hubble time $T_H = R(t)/\dot{R}(t)$ as well as being large compared to T_L.

The most convenient form of the field equations for the gravitational field $g_{\mu\nu}$ and the electromagnetic field $F^{\mu\nu}$ to use in constructing the EIH surface integrals is that given by Landau and Lifshitz [3] and they are of the form (I will use units in which $G = c = 1$, latin indices run from 1 to 3, greek indices run from 0 to 3, and I employ the Einstein summation convention and the comma notation to denote partial derivatives)

$$U^{\mu\nu\rho}_{,\rho} = \Theta^{\mu\nu} \tag{1}$$

where

$$U^{\mu\nu\rho} = -U^{\mu\rho\nu} = \frac{1}{16\pi}\{(-g)(\mathfrak{g}^{\mu\nu}\mathfrak{g}^{\rho\sigma} - \mathfrak{g}^{\mu\rho}\mathfrak{g}^{\nu\sigma})\}_{,\sigma}$$

$$\Theta^{\mu\nu} = (-g)(T^{\mu\nu} + t^{\mu\nu}_{\mathrm{LL}})$$

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and

$$(\sqrt{-g}\,F^{\mu\nu})_{,\nu} = 0 \quad \text{and} \quad F_{|\mu\nu,\rho|} = 0\,. \tag{2}$$

In these equations $g = \det(g_{\mu\nu})$, $\mathfrak{g}^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}$, $t_{\rm LL}^{\mu\nu}$ is the Landau-Lifshitz stress-energy pseudotensor, and $T^{\mu\nu}$ is the electromagnetic stress-energy tensor given by

$$T^{\mu\nu} = \frac{1}{16\pi}\,(g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - 4g_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma})\,.$$

(We do not include a matter stress-energy tensor in the expression for $\Theta^{\mu\nu}$ since the sources of the fields are assumed to be compact and to vanish on the EIH surfaces.)

Because of the antisymmetry of $F^{\mu\nu}$ and $U^{\mu\nu\rho}$ in their last two indices it follows that $F^{rs}_{,s}$ and $U^{\mu rs}_{,s}$ are three-dimensional curls whose integral over a closed surface vanishes. As a consequence, integration of Eq. (1) over a closed spatial surface in a t = const hypersurface gives

$$\oint (U^{\mu r0}_{,0} - \Theta^{\mu r})n_r\,dS = 0\,, \tag{3}$$

where n_r is a unit surface normal. In a similar way we get from Eq. (2) the result

$$\oint (\sqrt{-g}\,F^{r0})_{,0}n_r\,dS = 0\,. \tag{4}$$

It is these last two equations, when the surfaces surround a source point, that are used to obtain equations of motion for our system. They come from noting that, since these equations must hold on any two-surface, the sum of the contributions to the integrals that are surface independent must by themselves be zero. (The surface dependent terms will in all cases vanish as a consequence of the field equations.) This requirement then leads to restrictions on the motion of the sources comprising the system. In practice one chooses the surfaces to be small spheres centered on the sources and sets to zero those terms that are independent of the radii of the spheres. It should be noted that, since all integrals are over surfaces on which the fields and their derivatives are all finite, no infinite integrals appear that must be renormalized away.

To solve the field equations it proves convenient to perturb off of the metric densities $\mathfrak{g}^{\mu\nu}$ whose unperturbed components are given by

$$\mathfrak{g}^{\mu\nu} = \mathrm{diag}(R^3, -R, -R, -R)\,.$$

In addition we employ the method of multiple time scales [4]. To do so we assume that our fields depend on the time t through a dependence on $t_S = \varepsilon_S t$ and $t_H = \varepsilon_H t$, where $\varepsilon_S = T_L/T_S$ and $\varepsilon_H = T_L/T_H$, and expand them in powers of ε_S and ε_H. (In keeping with our original restrictions on our time scales we assume that $\varepsilon_H < \varepsilon_S \ll 1$.) Since we are only concerned here with the modifications in the Newtonian equations of motion for charged and uncharged sources we need only find $\mathfrak{g}^{00}$ to order ε_S^2 and $\mathfrak{g}^{0r}$ to order ε_S^3. Therefore we can set

$$\mathfrak{g}^{00} = R^3 + \varepsilon_S^2 h \tag{5}$$

and

$$\mathfrak{g}^{0r} = -R + \varepsilon_S^3 h^r \tag{6}$$

where R is a function of t_H alone while h and h^r are functions of both t_H and t_S as well as the comoving spatial coordinates x^1, x^2, and x^3. In what follows we will consider only uncharged sources and simply state the results for charged sources.

The field equations require that

$$\nabla^2 h = 0\,.$$

For spherical sources the solution has the form

$$h = 4\sum \frac{\tilde{m}_A}{r_A} \tag{7}$$

where the index A labels the particles in the system and the sum is over all **A**. The $\tilde{m}$'s appearing here are as yet undetermined functions of t_S and t_H and $\mathbf{r}_A = \mathbf{x} - \mathbf{x}_A$ where the $\mathbf{x}_A$ are the Ath particle's coordinates and are also functions of t_S and t_H. Inserting this solution into the surface integral of Eq. (3) with $\mu = 0$ yields the two conditions

$$\partial_{t_S}\tilde{m}_A = 0 \quad \text{and} \quad \partial_{t_H}\tilde{m}_A = 2\,\frac{\partial_{t_H}R}{R}\,\tilde{m}_A$$

implying that

$$\tilde{m}_A = m_A R^2,$$

where m_A is a constant which we take to be the mass of the Ath particle.

In order to determine the h^r appearing in Eq. (6) it is necessary to impose coordinate conditions. For convenience we take them to be of the form

$$\partial_{t_S}h + \nabla\cdot\mathbf{h} + 4\,\frac{\varepsilon_H}{\varepsilon_S}\sum R^2\,\frac{m_A}{r_A^3}\,\mathbf{r}_A\cdot\partial_{t_H}\mathbf{x}_A = 0\,,$$

which together with the field equations yields the result that

$$\nabla^2\mathbf{h} = \mathbf{0}\,.$$

These last two equations together determine the h^r to be

$$h^r = 4R^2\sum m_A\,\frac{1}{r_A}\left(\partial_{t_S}x_A^r + \frac{\varepsilon_H}{\varepsilon_S}\,\partial_{t_H}x_A^r\right). \tag{8}$$

The above result, together with the expression for h given in Eq. (7), is sufficient to evaluate the surface integral in

Eq. (3) with $\mu = r$. When the integrations are carried out and the terms that are independent of the surface radii are set equal to zero, one obtains the results that

$$m_A \partial_{t_S}^2 \mathbf{x}_A + 2 \frac{\varepsilon_H}{\varepsilon_S} \partial_{t_S} \partial_{t_H} m_A \mathbf{x} = -\frac{1}{R^3} \sum{}' \frac{m_A m_B}{r_{AB}^3} \mathbf{r}_{AB} - 2 \frac{\varepsilon_H}{\varepsilon_S} m_A \frac{\partial_{t_H} R}{R} \partial_{t_H} \mathbf{x}_A + \mathcal{O}(\varepsilon_S^2, \varepsilon_H^2) \qquad (9)$$

where $\mathbf{r}_{AB} = \mathbf{x}_A - \mathbf{x}_B$ and where the primed sum is over all B except $B = A$.

A similar development can be carried out for charged particles. Assuming the same temporal scale ordering as in the uncharged case one obtains from Eqs. (2) the equation for the scalar potential ϕ

$$\nabla^2 \phi = 0,$$

which for spherically symmetric sources has the solution

$$\phi = \sum \frac{\tilde{q}_A}{r_A}.$$

Inserting this expression for ϕ into the surface integral in Eq. (4) yields the result that

$$R\tilde{q}_A = q_A$$

where q_A is a constant, which we take to be the charge of the Ath particle. The resulting equation of motion that follows from Eq. (3) is the same as Eq. (9) except for the addition of a term on the right-hand side of the form

$$\frac{1}{R^3} \sum{}' \frac{q_A q_B}{r_{AB}^3} \mathbf{r}_{AB}.$$

We see from the above that the effect of expansion is twofold: It modifies the strength of both the Newtonian and the Coulomb inverse square laws and adds a kind of cosmic "drag" given by the second term in the right-hand side of Eq. (9). To see the effect of these two modifications consider the case of two uncharged particles in circular orbits about each other with $\varepsilon_H \ll \varepsilon_S$. In this case the equations of motion (9) yield the results that

$$r^3 \omega = \frac{\alpha}{R^3} \quad \text{and} \quad r\omega = \frac{\beta}{R}$$

where α and β are constants, r is the comoving coordinate distance between the masses, and ω is the orbital angular velocity. It follows that

$$\omega = \text{const} \quad \text{and} \quad Rr = \text{const}$$

with similar results holding for particles bound together electrically.

In the Einstein-deSitter universe the comoving coordinate differences between test bodies is constant, while we see that the coordinate radii of our bound systems decreases inversely with R to the order of accuracy of our approximation. Whether we choose to call Rr or r the size of such a system is a matter of definition, although it is customary to call Rr the size. What matters in the end is that the ratio of the comoving coordinate differences between test bodies and r is increasing with R. We cannot say if it is the universe that is expanding while our measuring sticks remain fixed in length or the universe is fixed but our measuring sticks are shrinking. Neither view can be ruled out by observation.

Since the frequencies of both bound systems, charged and uncharged, are constant to the order of our accuracy, they each can be used as clocks. Thus we see that "gravitational" and "electrical" clocks will both measure cosmic time. We expect of course that in higher orders of approximation these results will no longer hold. If $\varepsilon_H \sim \epsilon_S$, the two terms in Eq. (9) will be of the same order of magnitude as the other terms so that our clocks will not measure cosmic time, although they will remain synchronous and their sizes will change with time. In the higher so-called post-Newtonian approximations they will not even remain synchronous. What effect these modifications might have on the long time evolution of physical systems such as stars has still to be explored.

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