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Statistical-Mechanical Foundation of the Ubiquity of Lévy Distributions in Nature

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We show that the use of the recently proposed thermostatistics based on the generalized entropic form $S_q \equiv k(1 - \sum_i p_i^q)/(q-1)$ (where $q \in \boldsymbol{R}$, with $q = 1$ corresponding to the Boltzmann-Gibbs-Shannon entropy $-k\sum_i p_i \ln p_i$), together with the Lévy-Gnedenko generalization of the central limit theorem, provide a basic step towards the understanding of why Lévy distributions are ubiquitous in nature. A consistent experimental verification is proposed.

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The importance of Lévy distributions in physics and related areas has long been known [1,2]. However, the intensive search for both experimental and computational quantitative verifications of such laws in physical systems is relatively recent. As an illustration of the ubiquity of such distributions, we mention the interesting and successful direct verifications performed for, among others, CTAB micelles dissolved in salted water [3], chaotic transport in a laminar fluid flow of a water-glycerol mixture in a rapidly rotating annulus [4], subrecoil laser cooling [5], the analysis of heartbeat histograms in healthy individuals [6], particle chaotic dynamics along the stochastic web associated with a $d = 3$ Hamiltonian flow with hexagonal symmetry in a plane [7], conservative motion in a two-dimensional periodic potential [8], and a computer simulation of a leaky faucet [9]. Consistently, the general trend nowadays is to put *Lévy-type anomalous (super)diffusion* on a similar footing with *normal, Brownian-type, diffusion.* However, in what concerns our understanding of their statistical-mechanical foundations, the situation is vastly different for the two types of diffusion. Indeed while deep understanding of Brownian motion was basically achieved with Einstein's celebrated 1905 paper [10], the situation is much less clear for Lévy-type superdiffusion.

Before discussing the Lévy case, we reproduce here (for dimension $d = 1$) one of the most elegant manners of obtaining normal diffusion from fundamental thermostatistics. The Boltzmann-Gibbs-Shannon entropy associated with one diffusing particle (along the x axis) is given by

$$S_1[p] = -k \int dx\, p(x) \ln[\sigma p(x)] \tag{1}$$

(where the subscript 1 will soon become clear), and $\sigma > 0$ is a *finite* characteristic length of the problem. We wish to optimize $S_1[p]$ (because we wish, in fact, to optimize the likelihood function $W_1[p] \propto e^{S_1[p]/k}$; see Ref. [11]), subject to the constraints

$$\int dx\, p(x) = 1\,, \tag{2}$$

$$\langle x^2 \rangle_1 \equiv \int dx\, x^2 p(x) = \sigma^2. \tag{3}$$

Using Lagrange parameters, we immediately obtain the optimizing distribution

$$p_1(x) = e^{-\beta x^2}/Z_1\,, \tag{4}$$

with $Z_1 \equiv \int_{-\infty}^{\infty} dx\, e^{-\beta x^2} = (\pi/\beta)^{1/2}$. If we substitute this p_1 in place of p in Eq. (3), we get $1/kT \equiv \beta = 1/(2\sigma^2)$.

Now the distribution $p_1(x)$ is that of *one jump.* We are interested, however, in the *macroscopic* phenomenon associated with N jumps, with N possibly very large. The N-jump distribution is given by the N-fold convolution product $p_1(x,N) = p_1(x) * p_1(x) * \cdots * p_1(x)$. The replacement of distribution (4) into this product easily yields $p_1(x,N) = (\beta/\pi N)^{1/2} e^{-\beta x^2/N}$. We verify then that

$$p_1(x,N) = \frac{1}{N^{1/2}}\, p_1\left(\frac{x}{N^{1/2}}\right), \quad \text{for } N = 1,2,3,\ldots. \tag{5}$$

Finally it follows that $\langle x^2\rangle_1(N) \equiv \int dx\, x^2 p_1(x,N) = \frac{1}{2}kTN$ for $N = 1,2,3\ldots.$ Since $N = Dt$, where t is time and D^{-1} is a characteristic time of the problem, we recover Einstein's celebrated result $\langle x^2\rangle_1 = \frac{1}{2}DkTt$.

This is the basic calculation. The ubiquity (and robustness) of normal diffusion in nature comes, *within Boltzmann-Gibbs thermostatistics,* from the *central limit theorem,* which essentially states that the N-fold convolution product $p(x,N)$ associated with an *arbitrary* (even) distribution law with *finite* second moment Δ^G is given, for $N \rightarrow \infty$, by

$$p(x,N) \sim \frac{1}{N^{1/2}}\, G\left(\frac{x}{N^{1/2}}; \Delta^G\right), \quad \text{for } N \gg 1, \tag{6}$$

where $G(y;\Delta^G)$ is the (centered) Gaussian with the *same second moment* Δ^G (preserved under convolution). Equation (5) is but a particular realization of this fact.

The central point addressed in the present Letter is *what* must be modified in the above (beautiful) picture so that the macroscopic ($N \rightarrow \infty$) "attractors" become symmetric Lévy distributions $L_\gamma(y;\Delta^L)$, instead of Gaussians. [Here $0 \le \gamma < 2$ is the parameter that controls the asymptotic behavior of the distribution: we have $L_\gamma(y;\Delta^L) \sim \Delta^L/|y|^{1+\gamma}$ as $|y| \rightarrow \infty$.]

Montroll and Shlesinger [1] showed that this can be achieved if one optimizes the standard entropy (1) and maintains the constraint (2), but replaces (3) by the *ad hoc* constraint

$$\left\langle \ln \frac{1}{2\pi} \int_{-\infty}^{\infty} d\kappa\, e^{-i\kappa x} e^{-a|\kappa|^\gamma} \right\rangle_1 = \text{const}, \tag{7}$$

where $a > 0$. But they eventually considered this possibility an *unsatisfactory* one, for the complexity of Eq. (7) makes it an undesirable candidate for an *a priori* constraint.

A different way out of the difficulty was recently suggested by Alemany and Zanette [12]. We work along the same lines. We generalize the entropy (1) into Ref. [13]

$$S_q[p] = k\, \frac{1 - \int d(x/\sigma)\,[\sigma p(x)]^q}{q - 1} \qquad (q \in \boldsymbol{R}). \tag{8}$$

Using the asymptotics $[\sigma p(x)]^{q-1} \sim 1 + (q - 1)\ln[\sigma p(x)]$, we see that Eq. (8) yields the traditional entropy (1) in the limit $q \rightarrow 1$. This generalized entropic form has been used during recent years to discuss a wide range of problems, including long-range interactions [14], quantum groups [15], anomalous diffusion in porous-type media [16], $d = 2$ Euler turbulence [17], and simulated-annealing optimization techniques [18].

We wish to optimize $S_q[p]$ (because we wish to optimize the likelihood function $W_q[p] \propto \{1 + (1 - q)S_q[p]\}^{1/(1-q)}$; see Refs. [19,20]) with the norm constraint as given by Eq. (2), and constraint (3) generalized as follows:

$$\langle x^2\rangle_q \equiv \int d(x/\sigma)\, x^2[\sigma p(x)]^q = \sigma^2. \tag{9}$$

This specific constraint on the *q-expectation value* of x^2 has been shown to preserve, for all values of q, the Legendre structure and the stability of thermodynamics [13] (in particular, $1/T = \partial S_q/\partial U_q$, where U_q is the q-expectation value of the Hamiltonian, that is, the generalized internal energy), the Ehrenfest theorem [21], the Onsager reciprocity theorem [20], and other properties. By introducing Lagrange parameters, we easily perform the above optimization and obtain

$$p_q(x) = [1 - \beta(1 - q)x^2]^{1/(1-q)}/Z_q,$$

with $Z_q \equiv \int dx\,[1 - \beta(1 - q)x^2]^{1/(1-q)}$. More specifically, we recover Eq. (4) for $q = 1$. For $-\infty \le q < 1$

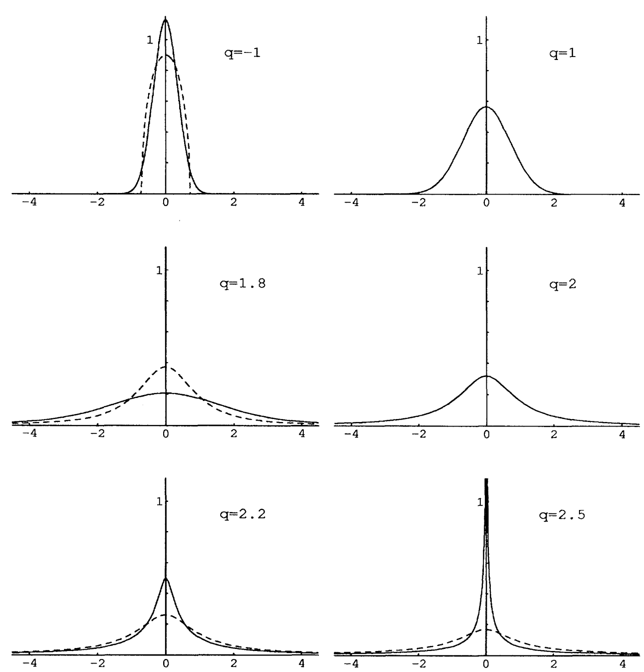

FIG. 1. Plot of the one-jump distribution (dashed lines $N = 1$) and N-fold convolution (solid lines $N \rightarrow \infty$). The latter is a Gaussian distribution for $q < \frac{5}{3}$ and a Lévy distribution for $q > \frac{5}{3}$. The dashed and solid curves coincide for $q = 1$ (Gaussian) and $q = 2$ (Lorentzian). Abcissas: $\beta^{1/2}x/N^{1/\gamma}$; ordinates: $N^{1/\gamma}p_q(x,N)/\beta^{1/2}$.

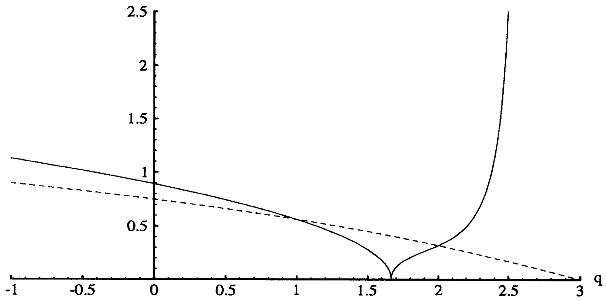

FIG. 2. Height at $x = 0$ of distributions of Fig. 1, for $-1 < q < 3$.

we obtain, setting $x_0 = [\beta(1 - q)]^{-1/2}$,

$$p_q(x) = \begin{cases} \frac{1}{x_0\sqrt{\pi}} \frac{\Gamma((5-3q)/2(1-q))}{\Gamma((2-q)/(1-q))} & \\ \quad \times[1 - (x/x_0)^2]^{1/(1-q)} & \text{if } |x| < x_0\,, \\ 0 & \text{otherwise.} \end{cases}$$

For $1 < q < 3$ we gave

$$p_q(x) = \left[\frac{\beta(q-1)}{\pi}\right]^{1/2} \frac{\Gamma(1/(q-1))}{\Gamma((3-q)/2(q-1))} \times \frac{1}{[1 + \beta(q-1)x^2]^{1/(q-1)}}\,.$$

Now this last expression behaves as $p_q(x) \propto x^{-2/(q-1)}$ as $|x| \rightarrow \infty$. For $q \geq 3$, the constraint (2) cannot be satisfied because Z_q diverges.

See Fig. 1 for typical examples of $p_q(x)$; the cases $p_{-\infty}$, p_1, p_2, and p_3 correspond, respectively, to Dirac's delta and the Gaussian, Lorentzian, and completely flat distributions. See also Fig. 2.

It can be easily verified that $\langle x^2\rangle_1(1) \equiv \int dx\, x^2 p_q(x)$ is *finite* for $q < \frac{5}{3}$ and *diverges* for $\frac{5}{3} \leq q \leq 3$, while $\langle x^2\rangle_q(1) \equiv \int d(x/\sigma)\, x^2[\sigma p_q(x)]^q$ is *finite* for all $q \leq 3$. The remarkable mathematical convenience of the q-expectation value is well illustrated with the Lorentzian (or Cauchy) distribution, $q = 2$. Indeed $\langle x^2\rangle_1(1) \propto \int_{-\infty}^{\infty} dx\, x^2(1 + \beta x^2)^{-1}$ *diverges* (and hence *is unacceptable as a constraint in any mathematically well-posed optimization procedure*), while $\langle x^2\rangle_2(1)$, which is proportional to $\int_{-\infty}^{\infty} dx\, x^2(1 + \beta x^2)^{-2}$, is *finite.*

Substitution of $p_q(x)$ in place of $p(x)$ in Eq. (9) yields $1/kT \equiv \beta = \Delta_q/\sigma^2$, with

$$\Delta_q = \begin{cases} \frac{1}{2}\left[\left(\frac{1-q}{2\pi}\right)^{1/2} \frac{\Gamma((5-3q)/2(1-q))}{\Gamma((2-q)/(1-q))}\right]^{2(q-1)/(3-q)} & \text{if } q < 1\,, \\ \frac{1}{2} & \text{if } q = 1\,, \\ \frac{1}{2}\left[\left(\frac{q-1}{2\pi}\right)^{1/2} \frac{\Gamma(1/(q-1))}{\Gamma((3-q)/2(q-1))}\right]^{2(q-1)/(3-q)} & \text{if } 1 < q \leq 3\,. \end{cases}$$

Through numerical and asymptotic analysis we verify that Δ_q vanishes for $q = -\infty$, increases monotonically with q until it achieves a maximum 0.79174 around $q = -0.63136$, then decreases monotonically, approaching 0 (as $[(3 - q)/4]^{4/(3-q)}$) when q approaches 3.

In general we have

$$\langle x^2\rangle_1(1) \equiv \int dx\, x^2 p_q(x) = \begin{cases} \frac{kT}{5-3q} & \text{if } -\infty \leq q < \frac{5}{3}\,, \\ \infty & \text{if } \frac{5}{3} < q \leq 3\,, \end{cases}$$

and

$$\langle x^2\rangle_q(1) \equiv \int d(x/\sigma)\, x^2[\sigma p_q(x)]^q = \Delta_q kT$$

for $-\infty \leq q \leq 3$, where we have used the explicit expressions of $p_q(x)$ obtained above.

Finally we are also going to need the Fourier transform $F_q(\kappa)$ of the distribution $p_q(x)$. Calculations in MATHEMATICA [22] show that $F_q(\kappa)$ has an analytic expression in terms of the modified Bessel function K. Its behavior, in the limit $\kappa \rightarrow 0$, is $F_q(\kappa) \sim e^{-\Lambda_q|\kappa|^\gamma}$, with

$$\gamma = \begin{cases} 2 & \text{if } -\infty \leq q \leq \frac{5}{3}\,, \\ (3 - q)/(q - 1) & \text{if } \frac{5}{3} < q < 3 \end{cases} \tag{10}$$

(which coincides with Ref. [12] for $q \geq \frac{5}{3}$) and

$$\Lambda_q = \begin{cases} \frac{1}{6}/(\frac{5}{3} - q) & \text{if } -\infty \leq q < \frac{5}{3}\,, \\ -\frac{\Gamma((q-3)/2(q-1))}{\Gamma((3-q)/2(q-1))}[4(q - 1)]^{-(3-q)/2(q-1)} & \text{if } \frac{5}{3} < q < 3\,. \end{cases} \tag{11}$$

We have $\Lambda_q \sim \frac{1}{6}/(q - \frac{5}{3})$ as $q \rightarrow \frac{5}{3} + 0$; moreover, $\Lambda_3 = 1$, and Λ_q has a flat minimum $\Lambda_q = 0.88954$ near $q = 2.3199$.

Now let us address the N-jump distribution $p_q(x,N)$, that is, the N-fold convolution product $p_q(x) * p_q(x) * \cdots * p_q(x)$. If we replace x by the scaled variable $x/N^{1/\gamma}$, the asymptotics at $\kappa = 0$ (and hence also Λ_q, for all q) are *invariant under convolution.* If $q < \frac{5}{3}$, the central limit theorem applies and we have, for $N \rightarrow \infty$, the Gaussian that has the *same second moment* as $p_q(x)$, namely,

$$p_q(x,N) \sim (\beta^{1/2}/N^{1/2})G(\beta^{1/2}x/N^{1/2}; 2\Lambda_q)\,.$$

Consequently,

$$\langle x^2\rangle_1(N) \equiv \int d(x/\sigma)\; x^2[\sigma p_q(x,N)] = D_q^G kTN\,,$$

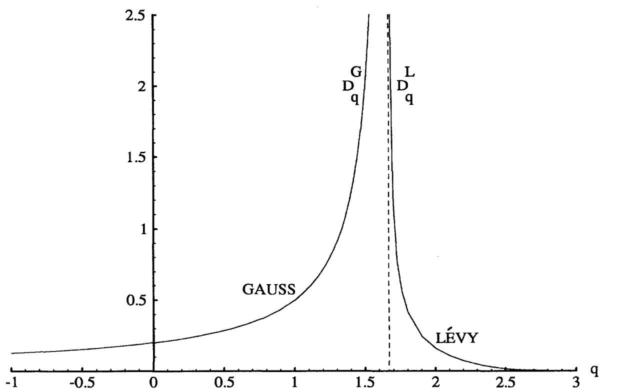

FIG. 3. Value of the (dimensionless) generalized diffusion coefficients D_q^G and D_q^L for $-1 < q < 3$. We have $D_q^L \sim 0.1211(q - \frac{5}{3})^{-2/3}$ as $q \to \frac{5}{3} + 0$.

with $D_q^G = 2\Lambda_q = \frac{1}{3}/(\frac{5}{3} - q)$. If $\frac{5}{3} < q < 3$, the Lévy-Gnedenko generalized central limit theorem applies [23] and we have, for $N \to \infty$, the Lévy distribution that has the *same long-distance asymptotic behavior* as $p_q(x)$ (in other words, the same short-κ behavior of their Fourier transforms). More specifically, as $N \to \infty$ we have

$$p_q(x,N) \sim \frac{\beta^{1/2}}{N^{1/\gamma}} L_\gamma\left(\frac{\beta^{1/2}x}{N^{1/\gamma}}; -\Lambda_q \frac{4^{(3-q)/2(q-1)}}{\sqrt{\pi}} \times \frac{\Gamma(1/(q-1))}{\Gamma((q-3)/2(q-1))}\right),$$

with γ and Λ_q given in Eqs. (10) and (11). Consequently,

$$\langle x^2 \rangle_q(N) \equiv \int d(x/\sigma)\, x^2[\sigma p_q(x,N)]^q = D_q^L kTN^{1/\gamma},$$

where D_q^L has been calculated numerically using MATHEMATICA and is shown in Fig. 3.

Summarizing, we see that, *within the present generalized thermostatistics,* the ubiquity and robustness of Lévy distributions in nature follow naturally from the *generalized central limit theorem.* In other words, the present approach only uses *simple a priori* constraints [(9) instead of (7)], thus satisfactorily accomplishing the Montroll and Shlesinger program [1]. *Normal, Gaussian-type, diffusion and the so-called anomalous, Lévy-type, superdiffusion are therefore unified in a single (and simple) picture.* This fact, added to various other satisfactory results [14–18], strongly supports the physical validity of the axiomatic Eqs. (8) and (9).

The experimental verification of the present framework could proceed as follows. In experiments, such as those of Ref. [3], controllable parameters, noted r (e.g., salinity or CTAB concentration in Ref. [3]), can exist which determine the type of diffusion. More precisely, there might exist r_c such that, for $r < r_c$, the system diffuses normally (hence $\gamma = 2$) and, for $r > r_c$, the system superdiffuses with $\gamma = \gamma(r) < 2$ (hence $q(r) = [3 + \gamma(r)]/[1 + \gamma(r)]$ for $d = 1$, and easily generalizable for arbitrary d). We expect the diffusion constant D_q^G associated with $\langle x^2 \rangle_1(N)$ to diverge when $r \to r_c - 0$, and the (generalized) diffusion constant D_q^L associated with $\langle x^2 \rangle_{q(r)}(N)$ to also diverge when $r \to r_c + 0$. If no correlation exists between successive jumps, the critical exponents associated with both divergences are expected to be 1 and $\frac{2}{3}$, respectively. Naturally the existence of correlations could modify these critical exponents.

Details on the present calculations will be published elsewhere.

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