PHYSICAL REVIEW LETTERS

VOLUME 75 | 17 JULY 1995 | NUMBER 3

Breathing Solitary Waves in a Sine-Gordon Two-Dimensional Lattice

J. M. Tamga,[1] M. Remoissenet,[1,*] and J. Pouget[2]

[1]*Laboratoire Université de Bourgogne, Phénomènes Non Linéaires, 6 Blvd Gabriel 21000 Dijon, France*
[2]*Laboratoire de Modèlisation en Mécanique, Université P. et M. Curie, 75252 Paris Cedex 05, France*

(Received 31 May 1994)

We study theoretically and numerically the dynamical behavior of a two-dimensional sine-Gordon lattice. We show that, via modulational instability, an initial-low-amplitude plane wave can evolve spontaneously into moving localized modes with large amplitude. These nonlinear modes, with dimensions depending on the characteristic wavelengths of the instability, behave like breathing solitary waves and present particlelike properties.

PACS numbers: 03.40.Kf, 63.20.Pw, 63.20.Ry

The concept of soliton has now become ubiquitous in modern sciences, and indeed can be found in various branches of physics and mathematics. In the strict sense, solitons appear in one- (space) dimensional systems, for which exact multisoliton solutions to various types of model field or lattice equations can be found. In two or three dimensions (2D or 3D), however, the situation becomes generally [1–3] more complicated; of various types of nonlinear localized modes (NLM) inherent to these situations vortices and spiral waves [1] are most typical; they play an important role in hydrodynamics, optics, condensed matter physics, and field theory. In this context the phenomenon of energy localization in nonlinear systems [4] is of intrinsic interest, and the existence of NLM can be a generic property of nonlinear Hamiltonian lattices [5]. Nevertheless, very little is known about the formation and behavior of such nonlinear entities in conservative systems. Thus, the question logically arises whether the remarkable soliton or solitonlike features also survive in conservative systems with higher dimensions.

To gain insight into this important problem, we have investigated the dynamical behavior of a 2D sine-Gordon lattice (2DSGL). For its continuum approximation solitonlike or vortexlike analytical solutions have been proposed [6–11], but few studies [12–14] have been devoted to the 2DSGL. The aim of the Letter is twofold. On the one hand, we study how a weak initial uniform perturbation can evolve spontaneously into NLM with large amplitude. On the other hand, we investigate the solitary-wave and particlelike properties of these robust nonlinear entities.

We consider an isotropic 2D planar model where rigid molecules rotate in the plane of a square lattice [15] of spacing a. Each rotator (molecule) with inertia J and angle of rotation $\Phi_{m,n}$ at site (m,n) interacts linearly with its first nearest neighbors and with a nonlinear periodic substrate potential. Here, G is the linear coupling coefficient and ω_0^2 is the strength of the potential barrier or square of the frequency of small oscillations in the bottom of the potential wells. The equation of motion of the rotator at site (m,n) is

$$\begin{aligned}\frac{d^2\Phi_{m,n}}{dt^2} = \frac{c_0^2}{a^2}[&(\Phi_{m+1,n} + \Phi_{m-1,n} - 2\Phi_{m,n}) \\ &+ (\Phi_{m,n+1} + \Phi_{m,n-1} - 2\Phi_{m,n})] \\ &+ \omega_0^2 \sin\Phi_{m,n}\,, \end{aligned} \tag{1}$$

where $c_0 = a\sqrt{G/J}$. If the rotation angle $\Phi_{m,n}$ varies slowly from one rotator to the next one, we can consider the long wavelength limit or continuum approximation (CA). In other terms the CA is valid [16] if the strength of the coupling is larger than the strength of the potential barrier; that is, if the discreteness parameter $c_0/\omega_0 = d \geq 2a$ or $3a$. Under this condition the CA of the discrete Frenkel-Kontorova (or 2DSGL) equations (1) is obtained by expanding in Taylor series $\Phi_{m,n}$ in terms of

 © 1995 The American Physical Society

its derivatives about the point ($x = ma$, $y = na$). Then, (1) is approximated by the 2D-SG equation

$$\Phi_{tt} - c_0^2(\Phi_{xx} + \Phi_{yy}) + \omega_0^2 \sin\Phi = 0, \qquad (2)$$

which describes the evolution of the scalar field $\Phi(x, y, t)$. Equation (2) will be used below.

In the weak amplitude limit we use the multiple scale perturbation technique [17,18]; that is, we consider a rapid carrier wave, with a slowly varying envelope $\psi(X, Y, T)$, propagating in the x direction: $\Phi_{m,n} = \varepsilon\psi(X, Y, \tau)\exp[i(\omega_c t - k_c na)] + \text{c.c.}$ Here, $X = \varepsilon x$, $Y = \varepsilon y$, and $\tau = \varepsilon^2 t$, where ε is a small parameter, $\varepsilon \ll 1$ represents the slow variables appropriate to the slow envelope variations; $\mathbf{k} = (k_c, 0)$ is the wave vector, ω_c is the circular frequency of the rapid carrier wave, and c.c. denotes the complex conjugate. Then, inserting the above expression into (1) we obtain [17–19] a 2D nonlinear Schrödinger (NLS) equation

$$i\,(\psi_\tau + V_g\psi_X) + (P_1\psi_{XX} + P_2\psi_{YY}) + Q|\psi|^2\psi = 0, \qquad (3)$$

where $V_g = (c_0^2/a\omega_c)\sin k_c a$, $P_1 = (c_0^2/2\omega_c)\cos k_c a$, $P_2 = c_0^2/2\omega_c$, and $Q = \omega_0^2/4\omega_c$ represent the group velocity in direction $0x$, the dispersion coefficients, and $\omega_c = [\omega_0^2 + 4c_0^2/a^2(\sin^2 k_c a/2)]^{1/2}$. A linear analysis of small perturbations [20,21] of the elementary plane wave solution of (3), $\psi = \psi_0\exp(iQ\psi_0^2\tau)$, with constant amplitude ψ_0, leads to a criterion for *modulation instability*: $P_1 q_L^2 + P_2 q_T^2 \leq 2Q\psi_0^2$. Here, q_L and q_T are the wave numbers of the perturbation in the longitudinal and transverse directions, respectively. Accordingly, an initial perturbation with a wave vector (q_L, q_T) satisfying the above relation can trigger instabilities in both directions of the lattice, with growth rate $\sigma^2 = (P_1 q_L^2 + P_2 q_T^2)(2Q\psi_0^2 - P_1 q_L^2 - P_2 q_T^2)$. The maximum instability occurs for grad$\sigma = 0$, that is, for $P_1 q_L^2 + P_2 q_T^2 = Q\psi_0^2$, and the corresponding growth rate is $\sigma_m = Q\psi_0^2$. In the special case of a unique longitudinal perturbation ($q_L \neq 0$, $q_T = 0$) or a unique transverse perturbation ($q_T \neq 0$, $q_L = 0$), the maximum instability occurs for $q_{Lm} = 2\pi/\lambda_{Lm} = \psi_0\sqrt{Q/P_1}$ or $q_{Tm} = 2\pi/\lambda_{Tm} = \psi_0\sqrt{Q/P_2}$, respectively. From these relations we obtain the ratio of the characteristic wavelengths λ_{Lm} and λ_{Tm},

$$\lambda_{Lm}/\lambda_{Tm} = \{\cos(k_c a) - (d/a)^2\sin^2(k_c a) \times [1 + 4(d/a)^2\sin^2(k_c a/2)]^{-1}\}^{1/2}, \qquad (4)$$

which depends on the discreteness parameter d and k_c the wave number of the carrier wave. As we shall see in the following, Eq. (4) will allow us to explain the geometry of the NLM. The above results tell us that, in the low amplitude limit, the 2DSGL Eq. (1) can be reduced to a 2DNLS Eq. (3), which is useful to predict instabilities, only, but not their evolution as time will increase. We note that, owing to the presence of the sine term in (1), these instabilities cannot evolve into a collapse as it could be obtained by performing a direct numerical study of (3).

Under these conditions by means of numerical simulations, we investigate the response of the lattice to an initial low amplitude plane wave modulated by a small perturbation. The simulations are performed (assuming $a = 1$) on lattice equations (1). We consider a lattice plane made of 106×82 points along with periodic boundary conditions, along x and y directions. The initial condition is provided by a harmonic carrier wave traveling in the x direction with small amplitude $2\psi_0 = 0.7$, wave vector $k_c = 0.18$, and frequency $\omega_c = 0.39$, which satisfies the dispersion relation $\omega_c = [\omega_0^2 + 4c_0^2/a^2(\sin^2 k_c a/2)]^{1/2}$, and velocity $v_c = \omega_c/k_c = 2.2$; we have chosen $\omega_0 = 0.3$. In order to trigger the instability of this wave, small ($\approx 10^{-3}\psi_0$) random perturbations (noise) are superposed to the initial velocity. Then, the system is isolated and let to evolve. A simulation for $d = 4.7$ shows that, as predicted by (3), the initial plane wave is modulationally unstable. When time further increases, the unstable state evolves into a localized NLM with large amplitude as shown at $t = 1500$ in Fig. 1(a). As time further evolves, the NLM moves freely as a whole along the x direction without spreading out, as represented in Fig. 1(c) for $t = 3110$. Inside the NLM we have $(\Phi_{i+1,j} - \Phi_{i,j})/a \leq 0.45$ rad and $(\Phi_{i,j+1} - \Phi_{i,j})/a \leq 0.35$ rad. These results suggest that the CA can be used to replace these finite differences by the gradient components $\partial\Phi/\partial x$ and $\partial\Phi/\partial y$. Thus for $d = 4.7$, to a first approximation, the lattice effects connected to a possible [22] Peierls-Nabarro barrier may be ignored and the NLM can be considered as a continuous entity as it will be assumed below when considering the motion of the center of mass of the NLM.

This NLM is *ellipse shaped* with the following dimensions: $L_x = 12.7$ and $L_y = 21$ at half width along the x and y directions. We note that the ratio $L_x/L_y = 0.60$ approaches the theoretical ratio $\lambda_{Lm}/\lambda_{Tm} = 0.76$ calculated from (4). Moreover, (4) shows that $L_x/L_y \to 1$ when dk_c becomes small; this suggests that in this case *circle-shaped* NLM should form. To check this expectation, keeping $d = 4.7$, we have performed numerical simulations for $k_c = 0.059$. As observed in Fig. 2, we obtain a NLM that approaches a circular shape: $L_x = 16$, $L_y = 18$, $L_x/L_y = 0.89$, and the agreement with $\lambda_{Lm}/\lambda_{Tm} = 0.96$ calculated from (4) is satisfactory. Moreover, simulations performed for $d = 2.1$, that is, close to the fully discrete regime, and $k_c = 0.18$ also show that circle-shaped NLM can emerge with $L_x = 7$, $L_y = 8$, and $L_x/L_y = 0.88$. In this case $\lambda_{Lm}/\lambda_{Tm} = 0.94$; again, the agreement is good. If the amplitude $2\psi_0$ of the initial plane wave is increased, the mean number of NLM's increases (not represented here). However, the NLM is deformed; this phenomenon is attributed to their mutual interactions which become important with their number, but will not be considered here.

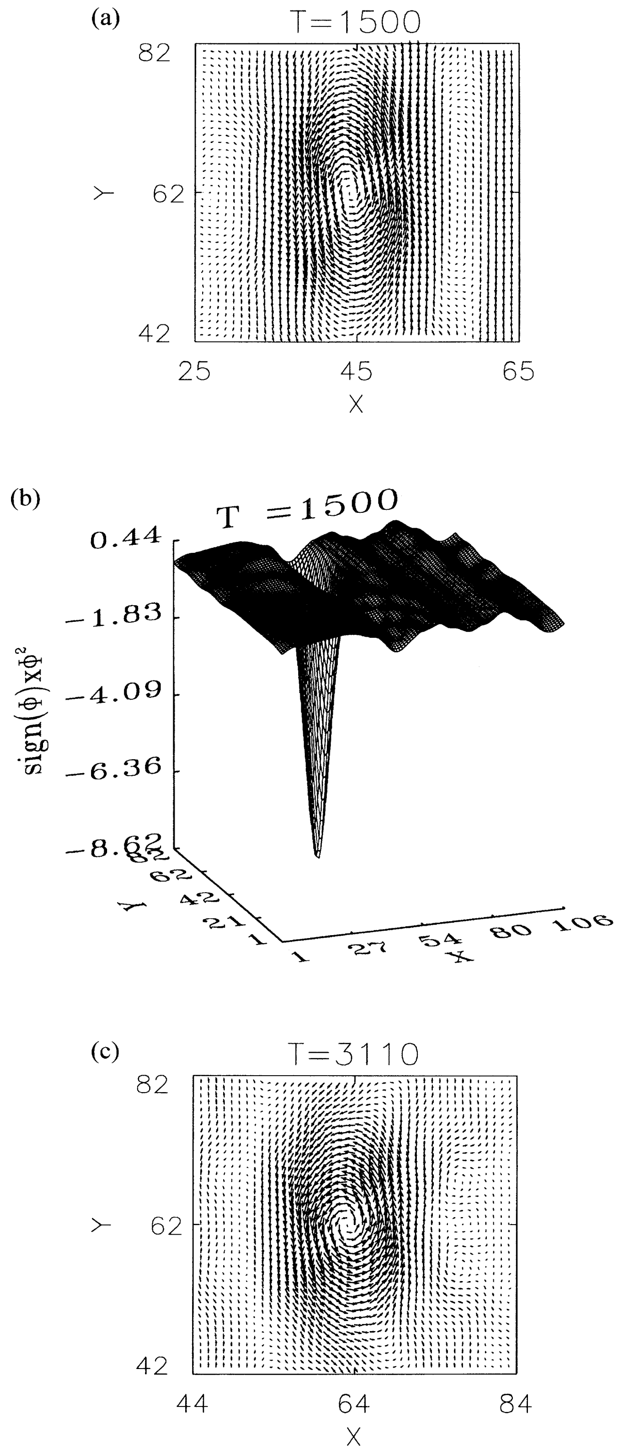

FIG. 1. (a) Representation at time $t = 1500$ of the NLM which emerges from the initial unstable plane wave, for $d = 4.7$ and $k_c = 0.18$. At each point (x, y), the arrow denotes a vector $\mathbf{u}(-\Phi_y, \Phi_x)$. Here, for convenience only, the region where the NLM is located is shown. (b) Representation of $\mathrm{sgn}(\Phi)\Phi^2$ which is proportional to the energy distribution in the lattice. (c) Representation of the NLM, later on, at time $t = 3110$.

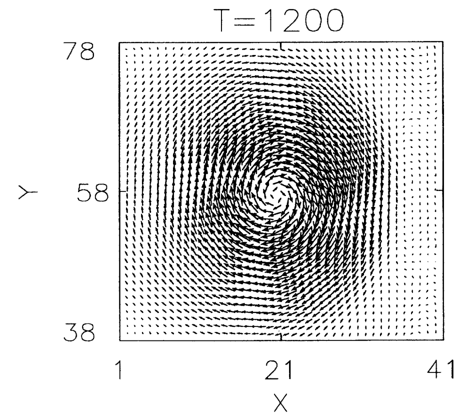

FIG. 2. For $d = 4.7$ and $k_c = 0.059$, an initial plane wave evolves into a circle-shaped NLM as depicted for $t = 1200$.

The fact that the NLM emerges spontaneously from an uniform initial state suggests that it is a stable nonlinear entity. In order to obtain more details about the dynamics of such a NLM, we have determined its profile in the x direction by representing (see Fig. 3) the quantity $|\Phi|^2 \,\mathrm{sgn}\Phi$ which corresponds to the energy distribution inside the NLM. The maximum amplitude, $A_m = 4.96$, is very large compared to the weak amplitude $2\psi_0 = 0.7$ of the initial extended perturbation. It presents an internal structure oscillating at frequency $\Omega_R = 0.15$ which lies below ($\Omega_R < \omega_0 = 0.3$) the phonon frequency band of the harmonic lattice described by (1) with $\sin\Phi_{m,n} \approx \Phi_{m,n}$. If one ignores this internal dynamics, the NLM moves as a whole with average velocity $v = 0.45$. These characteristic nonlinear signatures suggest that the NLM properties look like those of a 1D-SG *breather soliton.* In this context it is interesting to point out that the existence of breathers was recently proved for a broad range of Hamiltonian networks [23] of weakly coupled oscillators.

To further investigate the stability of the NLM *once it was formed,* we have switched the boundary conditions; that is, at time $t = 1500$ we have replaced the periodic

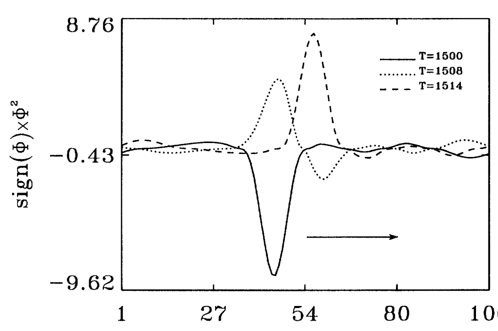

FIG. 3. Evolution of the oscillating (breathing) profile of the NLM as it propagates along the x direction.

boundary conditions along the x direction only by fixed boundary conditions at $x = 0$ and $x = 106$. Under these conditions, which correspond to an infinite potential well, the NLM that initially (at time $t = 1500$) moved from the left to the right is now reflected by the boundary at $x = 106$. Then, it reverses its sense of propagation, travels again, and reflects at the opposite boundary, $x = 0$, and so forth. These results confirm that the NLM is very robust. It is an oscillating concentration of energy that may be considered as a *nontopological solitary wave,* it behaves like an *extended* (elementary) *particle.* In this particular context we can consider, as seen above, that for d large the NLM behaves like a quasicontinuous entity. Under this approximation we also neglect the internal oscillations of the center of energy (mass). Thus, we analyze the dynamics of the NLM by using (2) and an extension [24] of the Ehrenfest theorem of quantum mechanics to nonlinear Klein-Gordon solitary waves. Thus, after some calculations we get the equation describing the motion of the "center of mass" G under the action of an external potential $V(x,y)$:

$$M(d^2\mathbf{OG}/dt^2) = -c_0^2\omega_0^2 \int (1 - \cos\phi)\, \mathrm{grad}V\, dx\, dy\,. \tag{5}$$

Here, $\mathbf{OG} = (X_G, Y_G)$ where O is the coordinates origin and M is the mass (energy) of the perturbed NLM:

$$M = \int \{\tfrac{1}{2}(\phi_t)^2 + (c_0^2/2)[(\phi_x)^2 + (\phi_y)^2] + \omega_0^2[1 + \mu V(1 - \cos\phi)]\}\, dx\, dy\,, \tag{6}$$

where $\mu \ll 1$. For the free motion ($V = 0$ everywhere, $Y_G = \text{const}$) with periodic boundary conditions, examined above, we find $dX_G/dt = v_G = 2.8$. For the back and fourth motion in an infinite potential well ($V = 0$ everywhere and $V = \infty$ at $x = 0$ and $x = 106$) along the x direction, also studied above, we have not detected energy radiation from the NLM. Like for the free motion, the time t_E, or so-called "Ehrenfest time" [25] (by analogy with quantum mechanics), over which the NLM could be delocalized is very long, say infinite. We have also studied the NLM motion in the presence of a harmonic potential $\mu V(x) = (\mu/2)(x - x_0)^2$, with $\mu = 0.004$, which at time $t = 4100$ replace the periodic boundaries in the x direction. The velocity of the center of mass dX_G/dt was calculated ($Y_G = \text{const}$) numerically and plotted in Fig. 4 as a function of time. Contrary to the square well potential case, both the amplitude and period of oscillations of G decrease with time. It means that with such a perturbating potential, which is nonzero everywhere except at center $x_0 = 53$, the NLM (quasiparticle) radiates its energy or mass during its motion and becomes delocalized over a time $t_E' \approx 800 \ll t_E$.

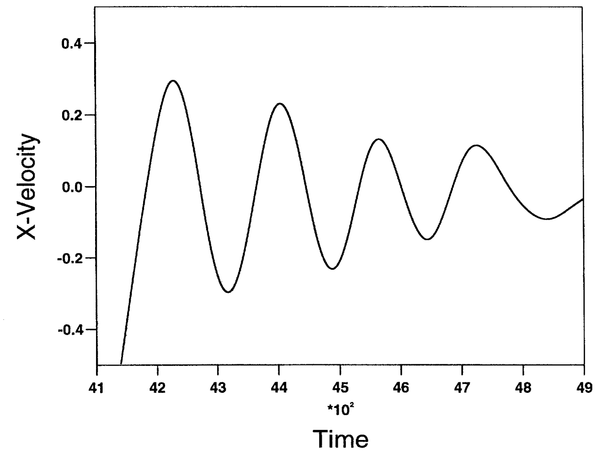

FIG. 4. Plot of the velocity dX_G/dt of the center of mass of the NLM (for $k_c = 0.18$ and $d = 4.7$), as a function of time t, as it moves in a harmonic potential well (see text). The amplitude and period of the oscillations decrease with time and the NLM becomes delocalized over a time $t_E' = 800$.

In conclusion, breathing NLM can form and propagate in a 2D sine-Gordon lattice. Such 2D solitary waves are very robust; they present properties of extended quasiparticles. Although until now we have been unable to find any approximate analytical solitons describing them, we think that such very long-lived NLM should play an important role in the various physical systems modeled by the 2D-discrete SG equation.

*Electronic address: remsnet@satie.u-bourgogne.fr

[1] M.V. Nezlin and E. Snezhkin, *Rossby Vortices, Structures, Solitons* (Springer-Verlag, Berlin, New York, 1993).
[2] V.G. Makhankov, *Soliton Phenomenology* (Kluwer, Dordrecht, The Netherlands, 1990).
[3] E.M. Maslov, Phys. Lett. A **151**, 47 (1990).
[4] A.J. Sievers and S. Takeno, Phys. Rev. Lett. **61**, 970 (1988).
[5] S. Flach, Phys. Rev. E **50**, 3134 (1994), and references therein.
[6] G. Grella and M. Marinaro, Lett. Nuovo Cimento **23**, 459 (1978).
[7] O. Hudak, Phys. Lett. **89A**, 245 (1982).
[8] A.B. Borisov, A.P. Tankeyev, A.G. Shagalov, and G.V. Bezmaternih, Phys. Lett. **11A**, 15 (1985).
[9] S. Takeno, Prog. Theor. Phys. **68**, 992 (1982).
[10] P.L. Christiansen and O.H. Olsen, Phys. Script. **20**, 531 (1979).
[11] N. Martinov and N. Vitanov, J. Phys. A **25**, 3609 (1992).
[12] F. Yoshida, F. Okwamoto, and T. Nakayama, J. Phys. Soc. Jpn. **51**, 1329 (1982).
[13] H. Kato, Y. Okwamoto, and T. Nakayama, J. Phys. Soc. Jpn. **52**, 1334 (1983).
[14] S. Takeno, K. Hori, K. Ohtsuka, and S. Homma, J. Phys. Soc. Jpn. **63**, 1295 (1994).
[15] J. Pouget, M. Remoissenet, and J.M. Tamga, Phys. Rev. B **47**, 14 866 (1993).
[16] M. Remoissenet, *Waves Called Solitons* (Springer-Verlag, Berlin, New York, 1994).

[17] T. Taniuti and N. Yajima, J. Math. Phys. (N.Y.) **10**, 1369 (1969).
[18] T. Kawahara, J. Phys. Soc. Jpn. **35**, 1537 (1973).
[19] M. Remoissenet, Phys. Rev. B **33**, 2386 (1986).
[20] D. U. Martin and H. C. Yen, Wave Motion **2**, 215 (1980).
[21] N. M. Akmediev, D. R. Heathley, G. I. Stegeman, and E. M. Wright, Phys. Rev. Lett. **65**, 1423 (1990).
[22] S. Flach and C. R. Willis, Phys. Rev. Lett. **72**, 177 (1994).
[23] R. S. MacKay and S. Aubry (to be published).
[24] G. Reinisch and J. C. Fernandez, Phys. Rev. Lett. **67**, 1968 (1991).
[25] E. J. Heller and S. Tomsovic, Phys. Today **46**, 38 (1993).