# Hidden Breit-Wigner Distribution and Other Properties of Random Matrices with Preferential Basis

Ph. Jacquod,[1] and D. L. Shepelyansky[2,*]

[1]*Institut de Physique, Université de Neuchâtel, 1, Rue A.L. Breguet, 2000 Neuchâtel, Switzerland*

[2]*Laboratoire de Physique Quantique, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse, France*

(Received 26 June 1995)

We study statistical properties of a class of band random matrices which naturally appears in systems of interacting particles. The local spectral density is shown to follow the Breit-Wigner distribution in both localized and delocalized regimes with width independent of the band or system size. We analyze the implications of this distribution to the inverse participation ratio, level spacing statistics, and the problem of two interacting particles in a random potential.

PACS numbers: 71.55.Jv, 05.45.+b, 72.10.Bg

Intensive investigations of band random matrices (BRM) have been done during the last few years [1,2]. Different regimes corresponding to localized and delocalized wave functions have been studied numerically and analytically, and it has been shown that the transition from one regime to another can be described by one scaling function depending on the ratio of the localization length in the infinite system $l \sim b^2$ to the size of the matrix $N$, where the parameter $b$ determines the size of the band $2b + 1$. Similar types of matrices appear in such physical systems as quasi-one-dimensional disordered wires and such models of quantum chaos such as the kicked rotator that gives additional grounds for investigation of BRM.

The above BRM can be also considered as a reasonable model of one-particle localization in a disordered wire of finite size [1,2]. However, recent investigations of two interacting particles (TIP) in a random potential [3] showed that another type of BRM naturally appears in interacting systems. Indeed for interacting particles there is one preferential basis which corresponds to eigenstates without interaction. In this basis, the total Hamiltonian is the sum of a diagonal matrix, with elements given by the sum of one-particle energies, and a BRM, which describes interaction induced transitions between eigenstates of the noninteracting problem. The first investigations of such superimposed BRM (SBRM) allowed one to find the dependence of the localization length $l_{sb}$ on the amplitude $W_b$ of large fluctuations on the diagonal and to obtain the localization length $l_c$ for two-particle coherent propagation in a random potential on a distance much larger than one-particle localization length $l_1$ [3].

While from the TIP model it is clear that matrices with preferential basis should describe interesting physical effects in interacting systems, only a few investigations in this direction have been done up to now [4,5]. In this paper we investigate the properties of such matrices, in particular, the local spectral density and the inverse participation ratio (IPR). Because of the close connection between the SBRM and the TIP problem, the obtained results can also be used for the latter case.

The matrix we study is the sum of a random diagonal matrix and a conventional BRM:

$$H_{n,n'} = \eta_n \delta_{n,n'} + \zeta_{n,n'}/\sqrt{2b+1}, \qquad (1)$$

with $-W_b \le \eta_n \le W_b$, $-1 \le \zeta_{n,n'} \le 1$ for $|n - n'| \le b$ and $\zeta_{n,n'} = 0$ elsewhere. The connection with the TIP is given by $b \sim l_1^2$ and $W_b \sim 4\sqrt{l_1}V/U$ in terms of the interaction strength $U$ and the one-particle energy bandwidth $4V$, $l_1 \gg 1$. This matrix describes a one-dimensional two-particle Anderson model, with on-site interaction $U$, in the basis of noninteracting eigenstates. In [3] it was shown that the eigenstates of (1) are localized with localization length $l_{sb} \approx b^2/2W_b^2$ for $1 < W_b \ll \sqrt{b}$. This leads to an enhancement of the length of coherent TIP propagation $l_c = l_{sb}/l_1 \sim l_1^2(U/V)^2/32$ independent on the sign of interaction.

Our numerical investigations of SBRM (1) show that, in addition to the standard exponentially localized form, the eigenstates are also characterized by large amplitude fluctuations of probability on nearby sites. A typical example of such an eigenstate is presented in Fig. 1. The spike eigenstate structure is clearly noticeable. This implies that only certain unperturbed states have strong admixtures into the given eigenstate. Such eigenstate structure is quite different from the case of conventional BRM. For a better understanding of these spiked fluctuations we study the local spectral density $\rho_W$ introduced by Wigner [6] and analyzed in BRM with a linearly growing diagonal corresponding to conservative systems [7,8]:

$$\rho_W(E - E_n) = \sum_\lambda |\psi_\lambda(n)|^2 \delta(E - E_\lambda). \qquad (2)$$

The function $\rho_W$ characterizes the average probability $P(|\psi_\lambda(n)|^2) = \rho_W(E - E_n)$ of eigenfunction $\psi_\lambda(n)$ on site $n$ with energy $E_n = H_{n,n}$, where $\lambda$ is the eigenvalue index and $n$ marks the original basis. Our numerical investigations in a wide range of parameters ($20 \le b \le$

0031-9007/95/75(19)/3501(4)$06.00  

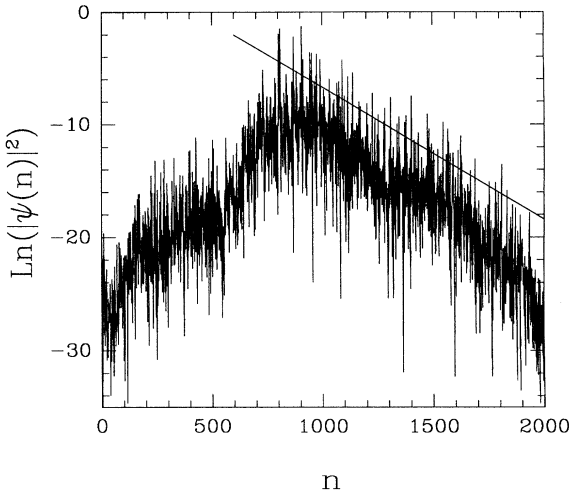


FIG. 1. Localized eigenfunction of a SBRM with $W_b = 7$, $b = 100$, and $N = 2001$. The solid line indicates the exponential localization with $l_{\rm sb} \approx 171$ in agreement with results obtained in [3], Eq. (3).

2000, $201 \leq N \leq 4001$, and $1.5 \leq W_b \leq 40$) in both localized ($l_{\rm sb} \ll N$) and delocalized ($l_{\rm sb} \gg N$) regimes show (see Fig. 2) that $\rho_W$ is well described by the well-known Breit-Wigner distribution $\rho_W = \rho_{\rm BW}$:

$$\rho_{\rm BW}(E - E_n) = \frac{\Gamma}{2\pi[(E - E_n)^2 + \Gamma^2/4]}, \qquad \Gamma = \frac{\pi}{3W_b}, \tag{3}$$

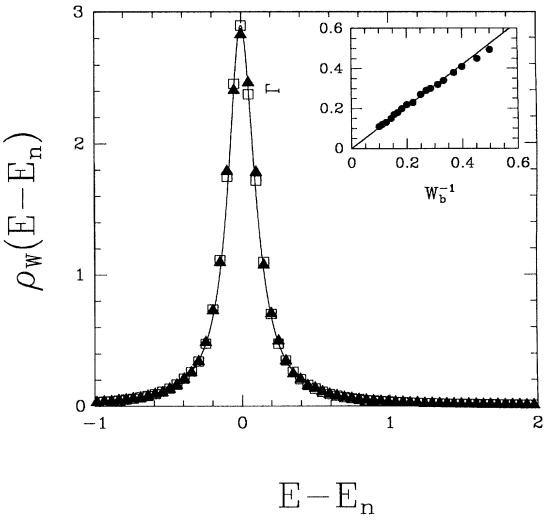


FIG. 2. Local spectral density which determines the average probability on a given site $P(|\psi_\lambda(n)|^2) = \rho_W(E - E_n)$ for $b = 100$, $W_b = 5$, $N = 201$ (triangles, 20 realizations of disorder), and $N = 2001$ (squares, 2 realizations of disorder). The solid line gives Breit-Wigner distribution (3) with $\Gamma = 0.21$. The inset shows the dependence of $\Gamma$ on $W_b^{-1}$: points are numerical data ($N = 1001, b = 100$); straight line is theory (3).

where $\Gamma$ is the distribution width. This distribution remains valid in localized and delocalized regimes under the condition that $\Gamma$ is much less than the energy width $\delta E \approx 1$ of $H_{n,n'}$ at $W_b = 0$. Usually the distribution $\rho_{\rm BW}$ appears in such physical systems as nuclei and complex atoms [8], where due to energy conservation the diagonal term $\eta_n$ grows linearly with $n$ that corresponds to a finite level density $\rho_E$. In this case the width is $\Gamma = 2\pi\rho_E\langle H^2_{n\neq n'}\rangle$ [6,8]. In our case for $W_b \gg 1$ all eigenenergies are homogeneously distributed in the finite interval $[-W_b, W_b]$, and for full matrices with $b = N/2$ we can use the above expression with $\rho_E = N/2W_b$, which gives $\Gamma$ in (3). For $b \ll N$ according to [3] one should replace $\rho_E$ by the density of directly coupled states $\rho_c = b/W_b$ that leads to the same expression for $\Gamma$. The theoretical formula for $\Gamma$, independent on $b$ and $N$, is in a good agreement with our numerical data (Fig. 2). The independence of $\Gamma$ on $b$ and $N$ makes our case quite different from the case of full matrix (1) studied before in [9].

For $W_b \gg 1$ the width of the Breit-Wigner peak is small and therefore according to (3) the probability on nearby levels is a strongly fluctuating spiked function. This spike structure of eigenfunctions can be characterized by the IPR $\xi_\lambda = (\sum_n |\psi_\lambda(n)|^4)^{-1}$, which counts the number of spikes independently on the distance between them. In the case of full matrices with $b = N/2$ the number of spikes can be estimated as the number of states in the interval $\Gamma$ that gives the average value of IPR $\xi = \langle 1/\xi_\lambda\rangle^{-1} \sim \rho_E\Gamma \approx N/2W_b^2$. The same estimate can be also used in the delocalized regime $l_{\rm sb} \gg N$ with $b \ll N$. Of course, this estimate is valid only when the number of states in the width $\Gamma$ is much larger than 1, that implies $\xi \gg 1$ or $W_b \ll \sqrt{N}$.

The numerical results for the dependence of IPR on $W_b$ in the delocalized regime are presented in Fig. 3. They demonstrate that for sufficiently large full matrices ($N = 4001$) this dependence approaches the above estimate. However, the convergence is rather slow so that for smaller $N$ values one has approximately $\xi \sim N/W_b^\alpha$, where the exponent $\alpha$ slowly changes with $N$. For example, $\alpha \approx 1.7$ for $N = 2001$. We attribute this very slow approach to the asymptotic value of $\alpha = 2$ to the quite restricted range of $W_b$ variation. Indeed on one side the width of the Breit-Wigner peak should not exceed the width of the energy band for $W_b = 0$ that gives $W_b \gg 1$. On the other side one should have $W_b \ll \sqrt{N}$. Another restriction appears for band matrices with $b < N/2$, namely, $l_{\rm sb} \gg N$. The data for this case (Fig. 3, full squares) show that for not very large $W_b$ the IPR is close to the regime of full matrices, while for large $W_b$ one enters the localized regime $l_{\rm sb} \ll N$, which should be studied separately.

It is interesting to note that in the delocalized regime even for $W_b \gg 1$ many levels are coupled by interaction if $\rho_E\Gamma \approx N/2W_b^2 \gg 1$. Therefore, one would expect that for $W_b < W_b^{\rm cr} \approx (N/2)^{1/2}$ the level spacing statistics

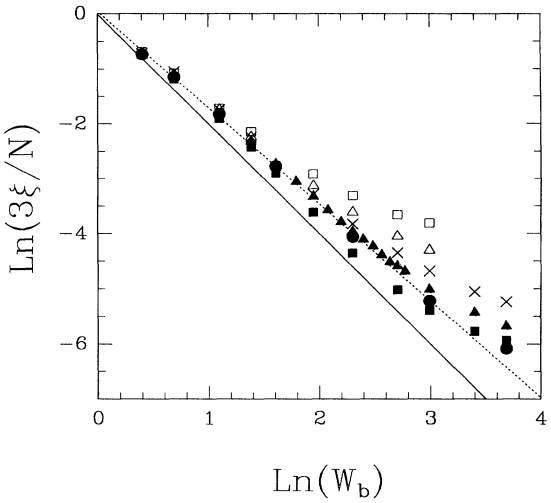


FIG. 3. IPR $\xi$ normalized with its limit value for the GOE case $N/3$ vs $W_b$ in the delocalized regime for $N = 2001$, $b = 300$ (full squares), and full matrices with $N = 251$ (open squares), $N = 501$ (open triangles), $N = 1001$ ($\times$), $N = 2001$ (full triangles), and $N = 4001$ (full circles). Dashed line shows the fit for full circles with $\alpha = 1.75 \pm 0.03$; solid lines shows theoretical slope $\alpha = 2$.

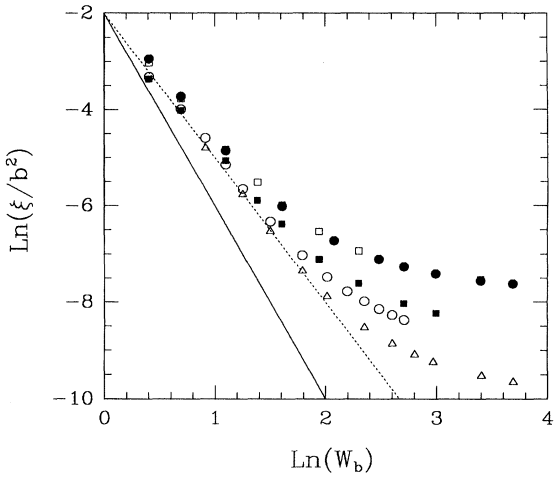


FIG. 4. Dependence of $\xi/b^2$ on $W_b$ in localized regime: $N = 2001$, $b = 50$ (full squares), and $b = 80$ (open squares); $N = 4001$, $b = 50$ (full circles), and $b = 100$ (open circles). Dashed line shows the slope from fit for open circles ($\beta = 3.0 \pm 0.1$) and solid line indicates theoretical slope $\beta = 4$.

$P(s)$ will be the same as in the case of Gaussian orthogonal ensemble (GOE) [10]. These expectations are not so evident, since the spiked structure of eigenfunctions apparently should lead to a decrease of overlapping matrix elements between eigenfunctions. However, our numerical results for matrices with $N \leq 8000$ show that $P(s)$ remains close to GOE for $1 < W_b < W_b^{\rm cr}$. They are also in agreement with the numerical results [11] for full matrices of smaller sizes showing that the transition border in $W_b$ between Poisson and GOE statistics scales as $N^{1/2}$. The question about other statistical properties of levels in the regime $1 < W_b < W_b^{\rm cr}$ remains open.

For the localized regime in the above estimate of $\xi$ one should replace $N$ by $l_{\rm sb}$, since only levels in the interval of one localization length can contribute to the IPR. This gives the expression

$$\xi \approx l_{\rm sb}/2W_b^{\beta-2} \approx b^2/4W_b^{\beta}, \quad \beta = 4, \qquad (4)$$

which is valid for $\xi \gg 1$ ($1 \ll W_b \ll \sqrt{b}$). The last condition together with $l_{\rm sb} \ll N$ gives strong restrictions for the numerical simulations ($1 \ll W_b \ll N^{1/4}$).

Our results for this localized case are presented in Fig. 4. The data can be empirically fitted by $\xi \sim b^2/W_b^{\beta}$ with $\beta \approx 3$, which differs from the theoretical value $\beta = 4$. We attribute this difference to the fact that we are not far enough in the asymptotic regime of large $b$ and $W_b$. Indeed for $W_b > b^{1/2}$ one enters in the perturbative regime, and the deviations from a power law become evident. We also checked that the probability distribution $P(|\psi_\lambda(n)|^4)$ is proportional to $\rho_{\rm BW}^2$ that gives additional grounds for the theoretical power $\beta = 4$. However, the simulations with large enough values of parameters $b, W_b$ require too large matrix sizes being beyond our numerical abilities. The numerically found value $\beta > 2$ implies that the number of peaks is smaller than the localization length $l_{\rm sb} \approx b^2/2W_b^2$, which determines the asymptotic exponential decay of the eigenstates. It would be desirable to have a more rigorous theoretical derivation of the IPR dependence on parameters in the localized regime.

The above results show that the SBRM (1) has many features similar to the photonic localization in a molecular quasicontinuum [12] as it was remarked in [3]. According to this analogy, the number of levels in one-photon transition (size) is of the order $b$, and the density of coupled states is $b/2W_b$. However, in the photonic model the levels are ordered in energy in a growing way that leads to a chain of equidistant Breit-Wigner peaks in an eigenstate [12]. For the SBRM (1) all levels are mixed in the energetic interval and the Breit-Wigner peak is hidden.

Let us now discuss the consequences of the result (4) for the TIP model. According to the relation between the parameters of SBRM and TIP given above, we obtain from (4) the expression for the IPR $\xi_c$ in the TIP model:

$$\xi_c \sim (U/V)^4 l_1^2 > 1. \qquad (5)$$

This result can be also derived directly from the density of states inside the localization length interval $l_c$ ($\rho_E \sim l_1 l_c/V$) and the transition rate $\Gamma_c \sim U^2/Vl_1$ obtained in [3]. Indeed, the number of levels in the Breit-Wigner peak is $\Gamma_c \rho_E \approx \xi_c$ that gives (5). This result shows that the number of noninteracting eigenstates $\xi_c$ contributing in the eigenfunction is quite large for $U \sim V$, while for $(U/V)^2 l_1 \ll 1$ this number is of the order of 1. However, the value of $\xi_c$ at $U \sim V$ is

much less than the number of unperturbed states $\Delta N$ contributing to the TIP eigenfunction in the unperturbed lattice basis. This number determines the IPR $\xi_{\max} \approx \Delta N \approx l_c l_1 \sim l_1^3$. The difference between $\xi_c$ and $\xi_{\max}$ shows that the noninteracting eigenbasis represents the real eigenfunctions in a much better way. It also stresses the fact that the IPR value is not basis invariant.

From the difference between $\xi_c$ and $\xi_{\max}$ it is possible to conclude that the coherent propagation of TIP goes by rare jumps of size $l_1$ between the states with approximately constant sums of noninteracting energies $E_s = \epsilon_n + \epsilon_{n'}$. The distribution over $E_s$ should have the Breit-Wigner form with the width $\Gamma_c$. The length of propagation by such jumps is $l_c \sim l_1^2 \gg l_1$. Because of this hidden Breit-Wigner distribution the IPR $\xi_c$ in the basis of noninteracting eigenstates is proportional to $l_1^2$ instead of "naive" $l_1^3$. For the case of TIP with $M$ transverse channels one should replace $l_1$ in (5) by $Ml_1$ with $l_1 \propto M$ being one-particle localization length.

If one-particle motion is ergodic in a $d$-dimensional system of size $L < l_1$, then its eigenfunction contains about $N_1 \approx L^d$ components. The matrix element of interaction is then $U_s \sim U/N_1^{3/2}$ [3], the density of coupled states $\rho_c \sim N_1^2/V$ and the Breit-Wigner width $\Gamma_c \sim U_s^2 \rho_c \sim U^2/N_1 V$ for $U < V$ is less than one-particle level spacing $\Delta_1 \approx V/N_1$. Therefore, it is possible that a concept of pairs formed by TIP can be useful even in the ergodic samples with $L < l_1$. In some sense, for $\Delta_2 \ll \Gamma_c \ll \Delta_1$, where $\Delta_2 \approx V/N_1^2$ is two-particle level spacing, one can at first average over fast one-particle motion and after that analyze the slow pair dynamics with typical time scale $1/\Gamma_c$. In the ergodic regime $L < l_1$ the IPR is $\xi_c \sim \Gamma_c \rho_c \sim N_1 (U/V)^2 \ll N_1^2$, and according to the above discussed properties of $P(s)$ in SBRM and the result [11] the GOE statistics for TIP should be observed for $\xi_c > 1$. For $L \gg l_1$ the strong enhancement of interaction [$\rho_c \Gamma_c \sim l_1^d (U/V)^2 \gg 1$] leads to delocalization of the TIP pairs in $d \geq 3$ below one-particle Anderson transition when noninteracting particles are well localized [13].

One of us (P. J.) gratefully acknowledges the hospitality of the Laboratoire de Physique Quantique, Université Paul Sabatier during his stay there, and another (D. L. S.) is grateful to the Yale University and the Godfrey Fund at University of New South Wales for hospitality during the process of work on the above problem. We thank Y. Alhassid, V. Flambaum, and O. Sushkov for useful discussions and remarks. This work is supported in part by the Fonds National Suisse de la Recherche.

*Also at Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia.

[1] G. Casati, L. Molinari, and F. M. Izrailev, Phys. Rev. Lett. **64**, 1851 (1990).
[2] Ya. V. Fyodorov and A. D. Mirlin, Phys. Rev. Lett. **67**, 2405 (1991); **71**, 412 (1993).
[3] D. L. Shepelyansky, Phys. Rev. Lett. **73**, 2607 (1994).
[4] G. Lenz and F. Haake, Phys. Rev. Lett. **67**, 1 (1991).
[5] J.-L. Pichard and B. Shapiro, J. Phys. **4**, 623 (1994); M. Kreynin and B. Shapiro, Phys. Rev. Lett. **74**, 4122 (1995).
[6] E. P. Wigner, Ann. Math. **62**, 548 (1955); **65**, 203 (1957).
[7] G. Casati, B. V. Chirikov, I. Guarneri, and F. M. Izrailev, Phys. Rev. E **48**, 1613 (1993); B. V. Chirikov (unpublished).
[8] V. V. Flambaum, A. A. Gribakina, G. F. Gribakin, and M. G. Kozlov, Phys. Rev. A **50**, 267 (1994).
[9] V. P. Tatarskii, Sov. Phys. Usp. **30**, 134 (1987); B. V. Chirikov, Report No. INP 87-123, Novosibirsk 1987.
[10] M. L. Metha, *Random Matrices* (Academic Press, New York, 1991).
[11] G. Lenz, K. Zyczkowski, and D. Saher, Phys. Rev. A **44**, 8043 (1991).
[12] D. L. Shepelyansky, Physica (Amsterdam) **28D**, 103 (1987).
[13] D. L. Shepelyansky (unpublished); Y. Imry (to be published); F. Borgonovi and D. L. Shepelyansky (to be published).