QCD Corrections from the Top Quark to Relations between Electroweak Parameters up to Order α_s^2

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The vacuum polarization functions $\Pi(q^2)$ of charged and neutral gauge bosons that arise from top and bottom quark loops lead to important shifts in relations between electroweak parameters that can be measured with ever-increasing precision. The large mass of the top quark allows approximation of these functions through the first two terms of an expansion in M_Z^2/M_t^2. The first three terms of the Taylor series of $\Pi(q^2)$ are evaluated analytically up to order α_s^2. Results for the subleading contributions to Δr and the effective mixing angle $\sin^2\overline{\Theta}$ are presented.

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The precision achieved in recent measurements of electroweak observables [1] has surpassed by far earlier expectations. The predictions for these quantities and the relations between them, which are based on the standard model (SM), are strongly affected by radiative corrections. Particularly important are those induced by virtual contributions from the heavy top quark [2]. Their verification provides an important test of the theory and its quantum corrections. The agreement between the value of M_t suggested by the CDF and D0 Collaborations [3] of $M_t = 176 \pm 8 \pm 10$ and $199^{+19}_{-21} \pm 22$ GeV, respectively, and deduced indirectly from precision measurements of $M_t = 173^{+12+18}_{-13-20}$ GeV [1], constitutes a triumph of the SM and a verification of the quantum corrections with increasing precision. A more refined understanding of these effects is on the agenda, including two- and even the dominant three-loop contributions [4,5].

As far as contributions from the top-bottom multiplet are concerned, perturbative results for self-energies $\Pi_{WW}(q^2)$, $\Pi_{ZZ}(q^2)$, $\Pi_{\gamma\gamma}(q^2)$, and $\Pi_{\gamma Z}(q^2)$ are available for arbitrary top and bottom masses up to order α_s [6–8].

This allows evaluation not only of the leading corrections, which are governed by the ρ parameter and increase with M_t^2, but also of the subleading terms. These are required for a complete calculation of Δr (entering the relation between G_F, M_W^2, M_Z^2, and α) or of the effective mixing angles $\sin^2\overline{\Theta}$ (governing asymmetries in Z production and decay). Recently also three-loop QCD corrections to the ρ parameter of $\mathcal{O}(G_F M_t^2 \alpha_s^2)$ have been calculated [4,5]. These, in turn, control the dominant terms of order $G_F M_t^2 \alpha_s^2$ in Δr and $\sin^2\overline{\Theta}$.

The technique described in [4,5] can be employed to obtain also the Taylor series coefficients of $\Pi(q^2)$ around $q^2 = 0$, in principle to arbitrary orders in q^2 and in second order in α_s. It will be demonstrated below that the two lowest terms in the expansion of Δr in M_Z^2/M_t^2 provide an excellent approximation to the full answer in one- and two-loop approximation, corresponding to α_s^0 and α_s^1 corrections. This justifies the expectation that also in order α_s^2 the leading terms $\propto G_F M_t^2 \alpha_s^2$ plus the subleading terms provide an adequate description of the complete answer for Δr and $\sin^2\overline{\Theta}$. This is verified by calculating the terms $\propto M_Z^2/M_t^2$, which indeed turn out to be negligible. The complete result for Δr and $\sin^2\overline{\Theta}$ to order α_s^2 is therefore at hand.

$M_W - M_Z$ connection and effective mixing angle.— It has become customary to express the magnitude of radiative corrections in the relation between M_W, M_Z, G_F, and α through the quantity Δr, defined through [9]

$$M_W^2 = \frac{M_Z^2}{2}\left(1 + \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}M_Z^2 G_F(1 - \Delta r)}}\right). \quad (1)$$

The influence of a heavy quark doublet can be expressed through transversal parts of gauge boson self-energies,

$$\Delta r_{tb} = \frac{c^2}{s^2}\,\mathrm{Re}\left(\frac{\Pi_{WW}(M_W^2)}{M_W^2} - \frac{\Pi_{ZZ}(M_Z^2)}{M_Z^2}\right) + \tilde{\Pi}_{\gamma\gamma}(0) + \frac{1}{M_W^2}[\Pi_{WW}(0) - \mathrm{Re}\Pi_{WW}(M_W^2)], \quad (2)$$

where $\tilde{\Pi}_{\gamma\gamma}(q^2) = \Pi_{\gamma\gamma}(q^2)/q^2$. In terms of the transversal parts of vector and axial current correlators Π^V and Π^A the building blocks for Δr are given by

$$\Pi_{WW} = (g^2/8)[\Pi^V(q^2, m_t, m_b) + \Pi^A(q^2, m_t, m_b)],$$

$$\Pi_{ZZ} = (g^2/16c^2)\sum_{i=t,b}[v_i^2 \Pi^{V,i}(q^2, m_i) + \Pi^{A,i}(q^2, m_i)] + (g^2/16c^2)\,\Pi^{A,S}(q^2, m_t, m_b), \quad (3)$$

$$\Pi_{\gamma\gamma} = g^2 s^2 \sum_{i=t,b} Q_i^2 \Pi^{V,i}(q^2, m_i),$$

$$\Pi_{\gamma,Z} = -(g^2 s/4c)\sum_{i=t,b} Q_i v_i \Pi^{V,i}(q^2, m_i),$$

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with $v_i = 2I_3^i - 4s^2 Q_i$. The sin and cos of the weak mixing angle are denoted by s and c. The "singlet" contribution to the axial current correlator $\Pi^{A,S}$, which originates from double triangle diagrams, occurs first in order α_s^2 and has been displayed separately.

Throughout this paper the mass of the bottom quark will be neglected. To circumvent the mass singularity that originates from the bottom loop contribution $\tilde{\Pi}_{\gamma\gamma}(0)$ is replaced by $\tilde{\Pi}_{\gamma\gamma}(q^2 = M_Z^2)$. The difference is accounted for by dispersion relations with input from the actual measurement of $\sigma(e^+e^- \to$ hadrons) in the low energy region [10]. So, we define $\Delta\bar{r}_{tb} \equiv \Delta r_{tb} - \tilde{\Pi}_{\gamma\gamma}(0) + \mathrm{Re}\tilde{\Pi}_{\gamma\gamma}(M_Z^2)$.

For diagrams involving a heavy top quark the approximation based on a Taylor series of $\Pi(q^2)$ around $q^2 = 0$, which is equivalent to an expansion in M_Z^2/M_t^2, leads to an adequate approximation. Even after charge and mass renormalization $\Pi(q^2)$ exhibits (in dimensional regularization) $1/\epsilon$ singularities, which cancel in the proper combination.

The reliability of the M_Z^2/M_t^2 expansion is demonstrated in Fig. 1 for the one- and two-loop results, respectively. The full answer (solid line) is compared to the approximation based on the term quadratic in M_t (dotted line) and the approximation including constant plus $\ln M_Z^2/M_t^2$ terms (dashed line). The latter provides an excellent approximation in the range $M_t > 150$ GeV; corrections of order $G_F M_Z^4/M_t^2$ may safely be ignored (dash-dotted line). Both the leading terms of the expansion and the full result are given in [7,11].

A second quantity of practical interest is the effective weak mixing angle, which governs, in particular, the asymmetries [12] in Z boson production and decay. It is related to $\sin^2\Theta \equiv 1 - M_W^2/M_Z^2$ through a correction factor $\sin^2\overline{\Theta} = [1 + \mathrm{Re}(\Delta\kappa)]\sin^2\Theta$, which in turn is influenced by the polarization functions of Eq. (3),

$$\Delta\kappa_{tb} = -\frac{c}{s}\frac{\Pi_{\gamma Z}(M_Z^2)}{M_Z^2} - \frac{c^2}{s^2}\mathrm{Re}\left(\frac{\Pi_{WW}(M_W^2)}{M_W^2} - \frac{\Pi_{ZZ}(M_Z^2)}{M_Z^2}\right).$$

The full analytic result and the approximation based on the M_Z^2/M_t^2 expansion are compared in Fig. 1. From this figure it is evident that (for $M_t \approx 180$ GeV) the next-to-leading corrections amount to about 25% of the $G_F M_t^2$ terms. The next-to-leading terms of order $G_F M_Z^2/M_t^2$ are below 1.2%. These considerations justify the restriction of α_s^2 corrections to the first two or at most three terms in the M_Z^2/M_t^2 expansion.

Three-loop corrections.—Three different types of integrals arise: The evaluation of the derivatives of $\Pi(q^2)$ resulting from diagrams where the massive top quark is coupled to the W or Z is reduced to the evaluation of tadpole integrals discussed in [4,5]. The derivatives are obtained through Taylor expansion of the respective integrands up to $\mathcal{O}(q^2)$ and projecting out the transverse part. Following [13,14] the resulting tadpole integrals are subsequently reduced to the master set listed in [4,5]. Diagrams involving external massless quark loops and internal top loops (and the anomaly graph) are treated with the large mass expansion technique [15]. The evaluation of $\Pi(q^2)$ up to three-loop diagrams involving massless diagrams only is performed with the FORM [16] package MINCER [17] implementing an algorithm developed in [13].

Setting $C_A = 3$, $C_F = 4/3$, and $\mu^2 = \overline{m}_t^2$ a fairly compact form for the subleading parts of $\Delta\bar{r}_{tb}$ and $\Delta\kappa_{tb}$ is obtained ($x_t = G_F\overline{m}_t^2/8\sqrt{2}\pi^2, \bar{l}_Z = \ln M_Z^2/\overline{m}_t^2$):

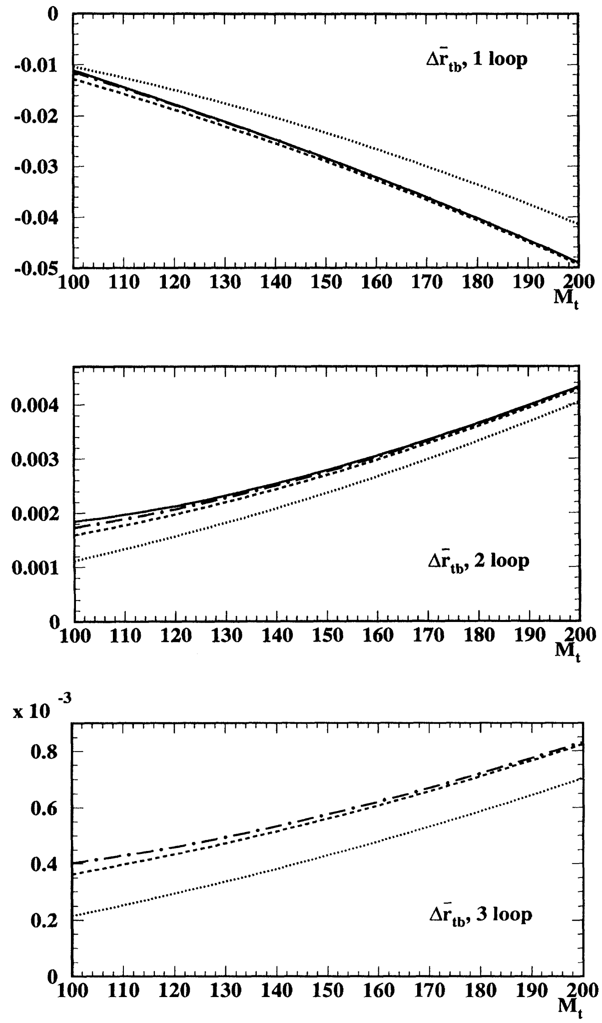

FIG. 1. $\Delta\bar{r}_{tb}$ as a function of M_t. The dotted, dashed, and dash-dotted curves correspond to an increasing number of terms in the approximation. Here a value $s^2 = 0.2321$ is chosen.

$$
\begin{aligned}
\Delta r_{tb}^{\overline{\rm MS}} = & -\frac{c^2}{s^2}\,\delta\rho_{\overline{\rm MS}} - 3\,\frac{c^2}{s^2}\,x_t\,\frac{M_Z^2}{\overline{m}_t^2}\Bigg\{\frac{1}{3} + \frac{8}{9}\,s^2\bar{l}_Z - \frac{16}{27}\,s^2 - \frac{2}{3}\,\bar{l}_Z \\
& + \frac{\alpha_s}{4\pi}\Bigg(\frac{88}{9} + \frac{128}{9}\,s^2\zeta(3) - \frac{128}{27}\,s^2\zeta(2) + \frac{32}{9}\,s^2\bar{l}_Z - \frac{496}{27}\,s^2 - \frac{32}{3}\,\zeta(3) + \frac{64}{27}\,\zeta(2) - \frac{8}{3}\,\bar{l}_Z\Bigg) \\
& + \left(\frac{\alpha_s}{4\pi}\right)^2\Bigg[n_f\Bigg(-\frac{4480}{243} + \frac{256}{27}\,s^2\zeta(3)\bar{l}_Z - \frac{2944}{81}\,s^2\zeta(3) + \frac{448}{27}\,s^2\zeta(2) - \frac{352}{27}\,s^2\bar{l}_Z + \frac{32}{27}\,s^2\bar{l}_Z^2 \\
& + \frac{2696}{243}\,s^2 - \frac{64}{9}\,\zeta(3)\bar{l}_Z + \frac{2080}{81}\,\zeta(3) - \frac{176}{27}\,\zeta(2) + \frac{88}{9}\,\bar{l}_Z - \frac{8}{9}\,\bar{l}_Z^2\Bigg) \\
& + \frac{94\,957}{243} + 856 S_2 s^2 - 428 S_2 + \frac{256}{81}\,D_3 s^2 - \frac{128}{81}\,D_3 - \frac{4480}{27}\,s^2\zeta(3)\bar{l}_Z + \frac{12\,016}{27}\,s^2\zeta(3) \\
& + \frac{12\,032}{81}\,s^2\zeta(4) - \frac{3200}{27}\,s^2\zeta(5) - \frac{52\,816}{243}\,s^2\zeta(2) - \frac{256}{27}\,s^2 B_4 + \frac{688}{3}\,s^2\bar{l}_Z - \frac{560}{27}\,s^2\bar{l}_Z^2 \\
& - \frac{24\,164}{81}\,s^2 + \frac{1120}{9}\,\zeta(3)\bar{l}_Z - \frac{8117}{18}\,\zeta(3) - \frac{6016}{81}\,\zeta(4) + \frac{800}{9}\,\zeta(5) \\
& + \frac{22\,736}{243}\,\zeta(2) + \frac{128}{27}\,B_4 - \frac{1400}{9}\,\bar{l}_Z + \frac{116}{9}\,\bar{l}_Z^2\Bigg]\Bigg\}, \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
{\rm Re}\left(\Delta\kappa_{tb}^{\overline{\rm MS}}\right) = & \frac{c^2}{s^2}\,\delta\rho_{\overline{\rm MS}} - 3\,\frac{c^2}{s^2}\,x_t\,\frac{M_Z^2}{\overline{m}_t^2}\Bigg\{-\frac{1}{3} - \frac{4}{9}\,s^2\bar{l}_Z + \frac{8}{27}s^2 + \frac{2}{3}\bar{l}_Z \\
& + \frac{\alpha_s}{4\pi}\Bigg(-\frac{88}{9} - \frac{64}{9}\,s^2\zeta(3) + \frac{64}{27}\,s^2\zeta(2) - \frac{16}{9}\,s^2\bar{l}_Z + \frac{248}{27}\,s^2 + \frac{32}{3}\,\zeta(3) - \frac{64}{27}\,\zeta(2) + \frac{8}{3}\,\bar{l}_Z\Bigg) \\
& + \left(\frac{\alpha_s}{4\pi}\right)^2\Bigg[n_f\Bigg(\frac{4480}{243} - \frac{128}{27}\,s^2\zeta(3)\bar{l}_Z + \frac{1472}{81}\,s^2\zeta(3) - \frac{224}{27}\,s^2\zeta(2) + \frac{176}{27}\,s^2\bar{l}_Z \\
& - \frac{16}{27}\,s^2\bar{l}_Z^2 - \frac{1348}{243}\,s^2 + \frac{64}{9}\zeta(3)\bar{l}_Z - \frac{2080}{81}\,\zeta(3) + \frac{176}{27}\,\zeta(2) - \frac{88}{9}\,\bar{l}_Z + \frac{8}{9}\,\bar{l}_Z^2\Bigg) \\
& - \frac{94\,957}{243} - 428 S_2 s^2 + 428 S_2 - \frac{128}{81}\,D_3 s^2 + \frac{128}{81}\,D_3 + \frac{2240}{27}\,s^2\zeta(3)\bar{l}_Z \\
& - \frac{6008}{27}\,s^2\zeta(3) - \frac{6016}{81}\,s^2\zeta(4) + \frac{1600}{27}\,s^2\zeta(5) + \frac{26\,408}{243}\,s^2\zeta(2) + \frac{128}{27}\,s^2 B_4 - \frac{344}{3}\,s^2\bar{l}_Z \\
& + \frac{280}{27}\,s^2\bar{l}_Z^2 + \frac{12\,082}{81}\,s^2 - \frac{1120}{9}\,\zeta(3)\bar{l}_Z + \frac{8117}{18}\,\zeta(3) + \frac{6016}{81}\,\zeta(4) \\
& - \frac{800}{9}\,\zeta(5) - \frac{22\,736}{243}\,\zeta(2) - \frac{128}{27}\,B_4 + \frac{1400}{9}\,\bar{l}_Z - \frac{116}{9}\,\bar{l}_Z^2\Bigg], \qquad (5)
\end{aligned}
$$

with [4] $B_4 = -1.762\,80\ldots$, $S_2 = 0.260\,434\ldots$, $D_3 = -3.027\,00\ldots$. The formula for $\delta\rho_{\overline{\rm MS}}$ can be found in [5].

This result is formulated in terms of the minimal substraction ($\overline{\rm MS}$) coupling α_s and the mass $\overline{m}_t$. Employing the two-loop relation between the $\overline{\rm MS}$ mass and the pole mass M_t [18] the $\overline{\rm MS}$ results are easily expressed in terms of M_t. The final result after setting $n_f = 6$ for Δr_{tb} is shown in Fig. 1. In this figure the terms of order $G_F\alpha_s^2 M_Z^4/M_t^2$ are also displayed. As shown in Fig. 1 and Table I their effect of the numerical result is extremely small and can safely be neglected.

In the lowest diagram of Fig. 1 the $\mathcal{O}(\alpha_s^2)$ corrections of $\Delta\overline{r}_{tb}$ and $\Delta\kappa_{tb}$ are presented as functions of M_t and the quality of the M_Z^2/M_t^2 expansion is confirmed. The difference between the quadratic term (dotted line) and the constant plus log term (dashed line) amounts to about 25%. Adding the subsequent term proportional M_Z^2/M_t^2 (dash-dotted line) barely affects the answer. The observation that the first few terms in the M_t^2 expansion provide an adequate approximation is nontrivial, and, in fact, is strongly tied to the large top mass. The approximation for Δk_{tb} is of equal quality.

The numerical effects on M_W and $\sin^2\overline{\Theta}$ are given in Table I. The numbers are obtained with the following input data: $\alpha_s(M_t^2) = 0.1092$ [corresponding $\alpha_s^{(5)}(M_Z^2) = 0.120$], $M_t = 175$ GeV, $M_Z = 91.188$ GeV, $M_H = 300$ GeV, $\alpha = 1/137.04$, and $G_F = 1.166\,39 \times 10^{-5}\ {\rm GeV}^{-2}$. In each column the terms of order α_s^0, α_s^1, and α_s^2 belonging to the corresponding expansion in the top mass are added up. For Δr in Eq. (1) we used $\Delta r = \Delta\alpha +$

TABLE I. Numerical results for M_t^2, the constant plus logarithmic, and $1/M_t^2$ contributions. For the numerical evaluation of $\delta \sin^2\overline{\Theta}/\sin^2\overline{\Theta}$ only the real part of $\Delta\kappa_{tb}$ is taken.

	M_t^2	M_t^2 + const	M_t^2 + const + $1/M_t^2$
$\delta M_W/M_W$ (OS)	0.006 77	0.008 38	0.008 25
$\delta \sin^2\overline{\Theta}/\sin^2\overline{\Theta}$ (OS)	−0.016 18	−0.018 32	−0.018 14
$\delta M_W/M_W$ ($\overline{\rm MS}$)	0.006 74	0.008 33	0.008 21
$\delta \sin^2\overline{\Theta}/\sin^2\overline{\Theta}$ ($\overline{\rm MS}$)	−0.016 10	−0.018 23	−0.018 04

TABLE II. The change in M_W separated according to powers in α_s and M_t in the on-shell scheme.

δM_W (MeV)	α_s^0	α_s^1	α_s^2
M_t^2	611.9	−61.3	−10.9
const	136.6	−6.0	−2.6
$1/M_t^2$	−9.0	−1.0	−0.2

$\Delta\overline{r}_{tb} + \delta_{\rm rem}$ with $\Delta\alpha = 0.059\,40$ [10]. $\delta_{\rm rem}$ contains all contributions of order α that are not present in the other two pieces and can, e.g., be found in [19]. In Table II the contributions are listed separately according to powers of α_s and M_t. One observes that the absolute prediction for the W mass is changed by −10.9 MeV if the $\alpha_s^2 M_t^2$ term is added to the full two-loop result. This increases to −13.7 MeV if also the constant and the $1/M_t^2$ suppressed terms are added. (For a fictitious top mass of 100 GeV the numbers would be −4.2 and −7.9 MeV, respectively.)

Conversely one may also study the shift in the prediction for M_t induced by these corrections. Neglect of the $\alpha_s^2 M_t^2$ term leads to an underestimate of M_t by 1.7 GeV; the correction calculated here induces an additional shift of 0.4 GeV. These effects should be compared to the anticipated uncertainty of indirect top mass determinations of 1–3 GeV through precision experiments plus radiative corrections (for fixed Higgs mass) and of 2 or perhaps even 1 GeV in production experiments at hadron or electron-positron colliders. Clearly the $\alpha_s^2 M_t^2$ term will be important for the analysis. The next term calculated here justifies the approximation inherent in this approach.

In summary, top mass dependent corrections to relations between electroweak parameters have been calculated up to order α_s^2, with the help of an expansion in M_Z^2/M_t^2. The quality of the approximation has been confirmed in one-, two-, and three-loop approximations.

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