Pair Production of Black Holes on Cosmic Strings

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We discuss the pair creation of black holes by the breaking of a cosmic string. We obtain an instanton describing this process from the C metric, and calculate its probability. This is very low for the strings that have been suggested for galaxy formation.

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The study of black hole pair creation has offered a number of exciting insights into the nature of quantum gravity, including some further evidence that the exponential of the black hole entropy really corresponds to the number of quantum states of the black hole [1–4]. Black hole pair production is a tunneling process, so it can be studied by finding a suitable instanton, that is, a Euclidean solution that interpolates between the states before and after the pair of black holes are created. The amplitude for pair creation is then given by e^{-I_i}, where I_i is the action of the instanton. Black hole pair creation has been commonly studied in the context of the Ernst metric [1,5], which describes the creation of a pair of charged black holes by a background electromagnetic field. The Lorentzian section of the Ernst metric represents a pair of charged black holes being uniformly accelerated by a background electromagnetic field.

If we consider the Ernst metric with zero background field, we obtain a simpler solution called the C metric [6]. The Lorentzian section still describes a pair of black holes uniformly accelerating away from each other, but there is now no background field to provide the acceleration. This means that there is either a conical deficit extending from each black hole to infinity, or a conical surplus running between the two black holes. These can be thought of, respectively, as "strings" pulling the two black holes apart, or a "rod" pushing them apart.

The purpose of this Letter is to argue that the C metric can also be interpreted as representing pair creation. Specifically, we can imagine replacing the conical deficit in the C metric with a cosmic string [7]. The Lorentzian section would then be interpreted as representing a pair of black holes at the ends of two pieces of cosmic string, being accelerated away from each other by the string tension. The Euclidean section of the C metric thus gives an instanton describing the breaking of a cosmic string, with a pair of black holes being produced at the terminal points of the string. The infinite acceleration, zero black hole mass limit of this breaking has been previously considered in [8]. We will calculate the action of the Euclidean C metric relative to flat space with a conical deficit, which gives the approximate rate for cosmic strings to break by this process. A similar calculation has previously been done for the breaking of a string with monopoles produced on the free ends [9], and we will show that our results agree with those, in the appropriate limit.

The charged C metric solution is

$$ds^2 = A^{-2}(x - y)^{-2}[G(y)\,dt^2 - G^{-1}(y)\,dy^2 + G^{-1}(x)\,dx^2 + G(x)\,d\varphi^2], \tag{1}$$

where

$$G(\xi) = (1 - r_- A\xi)(1 - \xi^2 - r_+ A\xi^3) \tag{2}$$

while the gauge potential is

$$A_\varphi = q(x - \xi_3), \tag{3}$$

where $q^2 = r_+ r_-$, and we define $m = (r_+ + r_-)/2$. We will only consider this magnetically charged case. We constrain the parameters so that $G(\xi)$ has four roots, which we label by $\xi_1 \le \xi_2 < \xi_3 < \xi_4$. To obtain the right signature, we restrict x to $\xi_3 \le x \le \xi_4$, and y to $-\infty < y \le x$. The inner black hole horizon lies at $y = \xi_1$, the outer black hole horizon at $y = \xi_2$, and the acceleration horizon at $y = \xi_3$. The axis $x = \xi_4$ points towards the other black hole, and the axis $x = \xi_3$ points towards infinity. To avoid having a conical singularity between the two black holes, we choose

$$\Delta\varphi = \frac{4\pi}{|G'(\xi_4)|}, \tag{4}$$

which implies that there will be a conical deficit along $x = \xi_3$, with deficit angle

$$\delta = 2\pi\left(1 - \left|\frac{G'(\xi_3)}{G'(\xi_4)}\right|\right). \tag{5}$$

Physically, we imagine that this represents a cosmic string of mass per unit length $\mu = \delta/8\pi$ along $x = \xi_3$. At large spatial distances, that is, as $x, y \longrightarrow \xi_3$, the C metric (1) reduces to flat space with conical deficit δ in accelerated coordinates. If we converted to cylindrical coordinates (t, z, ρ, φ) on the flat space, the acceleration horizon would correspond to the surface $z = 0$.

We might wonder whether it is possible to replace the conical singularity in the C metric with a real cosmic string. There are two potential problems: first of all, we have to be concerned about the effect of the string stress energy on the geometry in the neighborhood of the black hole horizon. However, it was shown in [7] that real vortices could pierce the black hole event horizon, so we will assume that this does not prevent the replacement.

Secondly, we might worry about having a string end in a black hole. If the strings are topologically unstable (that is, there are monopoles present before the phase transition at which the strings form), then we know that the strings can end at monopoles. But away from the event horizon, the field around a charged black hole is very similar to that around a monopole. It therefore seems reasonable to expect that a string can end in a black hole. It has been argued that any cosmic string can end on a black hole, even if the string is topologically stable [7] (this argument is also given in [10]). However, Preskill has remarked [11] that strings which are potentially the boundaries of domain walls *cannot* end on black holes, as the boundary of a boundary is zero (this category includes topologically stable global strings). For strings that cannot be the boundaries of domain walls, however, the argument of [7] applies (contrary to the statements in an earlier version of this paper).

We can obtain the Euclidean section of the C metric by setting $t = i\tau$ in (1). To make the Euclidean metric positive definite, we need to restrict the range of y to $\xi_2 \leq y \leq \xi_3$. There are then potentially conical singularities at $y = \xi_2$ and $y = \xi_3$, which have to be eliminated. We can avoid having a conical singularity at $y = \xi_3$ by taking τ to be periodic with period

$$\Delta\tau = \beta = \frac{4\pi}{G'(\xi_3)}. \tag{6}$$

If we assume that the black holes are extreme, that is, $\xi_1 = \xi_2$, then the spatial distance from any other point to $y = \xi_2$ is infinite, and $\xi_2 < y \leq \xi_3$ on the Euclidean section, so the conical singularity at $y = \xi_2$ is not part of the Euclidean section. Alternatively, if we assume $\xi_1 < \xi_2$, we can avoid having a conical singularity at $y = \xi_2$ by taking the two horizons to have the same temperature, so that both conical singularities can be removed by the same choice of $\Delta\tau$. This implies

$$\xi_2 - \xi_1 = \xi_4 - \xi_3. \tag{7}$$

As in the Ernst case, the former solution has topology $S^2 \times R^2 - \{pt\}$, while the latter has topology $S^2 \times S^2 - \{pt\}$.

We can obtain an instanton by slicing the Euclidean section in half along a surface $\tau = 0, \beta/2$. This instanton will interpolate between a slice of flat space with a conical deficit and a slice of the C metric, that is, a slice containing two black holes with conical deficits running between the black holes and infinity. Thus, this instanton can be used to model the breaking of a long piece of cosmic string, with oppositely charged black holes being created at the free ends. If $\xi_2 = \xi_1$, the black holes are extreme, while if $\xi_2 - \xi_1 = \xi_4 - \xi_3$, the black holes are nonextreme.

The semiclassical approximation to the amplitude for the string to break (per unit length per unit time) will be given by e^{-I_i}, where I_i is the action of this instanton. Using the fact that the extrinsic curvature of the slice $\tau = 0, \beta/2$ vanishes, we can show that the probability for the string to break is e^{-I_E}, where I_E is now the action of the whole Euclidean solution [12].

We will calculate the action of the Euclidean section following the technique used in [4,13]. In fact, the calculation is very similar to the calculation of the action in [4]. Since the solution is static, the action can be written in the form

$$I_E = \beta H - \tfrac{1}{4}\Delta\mathcal{A} \tag{8}$$

in the extreme case, and

$$I_E = \beta H - \tfrac{1}{4}(\Delta\mathcal{A} + \mathcal{A}_{\rm bh}) \tag{9}$$

in the nonextreme case, where the Hamiltonian is

$$H = \int_\Sigma N\mathcal{H} - \frac{1}{8\pi}\int_{S^2_\infty} N({}^2K - {}^2K_0), \tag{10}$$

$\Delta\mathcal{A}$ is the difference in area of the acceleration horizon, $\mathcal{A}_{\rm bh}$ is the area of the black hole event horizon, Σ is a surface of constant τ, and S^2_∞ is its boundary at infinity.

Since the volume term in the Hamiltonian is proportional to the constraint $\mathcal{H}$, which vanishes on solutions of the equations of motion, the Hamiltonian is just given by the surface term. In the surface term, ${}^2K$ is the extrinsic curvature of the surface embedded in the C metric, while ${}^2K_0$ is the extrinsic curvature of the surface embedded in the background, flat space with a conical deficit. We actually take a boundary "near infinity," and then take the limit as it tends to infinity after calculating the Hamiltonian. We choose the boundary in the C metric to be at $x - y = \epsilon_c$.

We want to ensure that we take the same boundary in calculating the two components of the Hamiltonian, which is achieved by requiring that the intrinsic metric on the boundary as embedded in the two spacetimes agree. We therefore want to write the flat background metric in a coordinate system that makes it easy to compare it to the C metric. We can, in fact, write the flat metric as

$$ds^2 = \bar{A}^{-2}(x - y)^{-2}[(1 - y^2)\,dt^2 - (1 - y^2)^{-1}dy^2 + (1 - x^2)^{-1}dx^2 + (1 - x^2)\,d\varphi^2], \tag{11}$$

where $\Delta\varphi = 2\pi - \delta$. Note that $\bar{A}$ represents a freedom in the choice of coordinates, and x is restricted to $-1 \leq x \leq 1$. A suitable background for the action calculation can be obtained by taking $t = i\tau$ and $y \leq -1$ in (11). We now take the boundary in the flat metric (11) to lie at $x - y = \epsilon_f$. It is easy to see that the induced metrics on the boundary will agree if we take

$$\bar{A}^2 = -\frac{G'(\xi_3)^2}{2G''(\xi_3)}A^2, \qquad \epsilon_f = -\frac{G''(\xi_3)}{G'(\xi_3)}\epsilon_c. \tag{12}$$

We can now calculate the two contributions to the Hamiltonian: the contribution from the C metric is (neglecting terms of order ϵ_c and higher)

$$\int_{S_\infty^2} N^2 K = \frac{8\pi}{A^2 \epsilon_c |G'(\xi_4)|} \left[1 - \frac{1}{4} \epsilon_c \frac{G''(\xi_3)}{G'(\xi_3)} \right], \quad (13)$$

while the contribution from the flat background is

$$\int_{S_\infty^2} N^2 K_0 = \frac{4\pi}{A^2 \epsilon_f} \left| \frac{G'(\xi_3)}{G'(\xi_4)} \right| \left(1 + \frac{1}{4} \epsilon_f \right). \quad (14)$$

Using (12), we see that these two surface terms are equal to this order. Thus, in the limit $\epsilon \longrightarrow 0$, the Hamiltonian vanishes.

Thus, the action is just given by

$$I_E = -\tfrac{1}{4} \Delta \mathcal{A} \quad (15)$$

in the extreme case and

$$I_E = -\frac{1}{4}(\Delta \mathcal{A} + \mathcal{A}_{\rm bh})$$

in the nonextreme case. Note that, as in the Ernst case [4], the probability to produce a pair of extreme black holes when the string breaks is suppressed relative to the probability to produce a pair of nonextreme black holes by a factor of $e^{\mathcal{A}_{\rm bh}/4}$.

The area of the black hole horizon is

$$\mathcal{A}_{\rm bh} = \int_{y=\xi_2} \sqrt{g_{xx} g_{\varphi\varphi}}\, dx\, d\varphi = \frac{4\pi(\xi_4 - \xi_3)}{A^2 |G'(\xi_4)| (\xi_3 - \xi_2)(\xi_4 - \xi_2)}. \quad (17)$$

To calculate the difference in area of the acceleration horizon, we calculate the area inside a circle at large radius in both the C metric and the background, and take the difference. The area of the acceleration horizon $y = \xi_2$ inside a circle at $x = \xi_3 + \epsilon_c$ in the C metric is

$$\mathcal{A}_c = \int_{y=\xi_3} \sqrt{g_{xx} g_{\varphi\varphi}}\, dx\, d\varphi = -\frac{\Delta\varphi}{A^2(\xi_4 - \xi_3)} + \frac{\Delta\varphi}{A^2 \epsilon_c} = -\frac{4\pi}{A^2 |G'(\xi_4)| (\xi_4 - \xi_3)} + \pi \rho_c^2 \left| \frac{G'(\xi_3)}{G'(\xi_4)} \right|, \quad (18)$$

where $\rho_c^2 = 4/A^2 G'(\xi_3) \epsilon_c$. The area of the acceleration horizon $z = 0$ inside a circle at $\rho = \rho_f$ in the flat background is

$$\mathcal{A}_f = \int \sqrt{g_{\rho\rho} g_{\varphi\varphi}}\, d\rho\, d\varphi = \pi \rho_f^2 \left| \frac{G'(\xi_3)}{G'(\xi_4)} \right|. \quad (19)$$

To ensure that we are using the same boundary in calculating these two components, we require that the proper length of the boundary be the same. This gives

$$\rho_f = \rho_c \left[1 + \frac{G''(\xi_3)}{G'(\xi_3)^2 A^2 \rho_c^2} \right]. \quad (20)$$

We can now calculate the difference in area; it is

$$\Delta \mathcal{A} = \mathcal{A}_c - \mathcal{A}_f = -\frac{4\pi}{A^2 |G'(\xi_4)|} \left[\frac{1}{(\xi_4 - \xi_3)} + \frac{G''(\xi_3)}{2G'(\xi_3)} \right] = -\frac{4\pi}{A^2 |G'(\xi_4)|} \left[\frac{2}{(\xi_3 - \xi_1)} + \frac{(\xi_2 - \xi_1)}{(\xi_3 - \xi_2)(\xi_3 - \xi_1)} \right]. \quad (21)$$

In the extreme case, $\xi_2 = \xi_1$, so the action is

$$I_E = -\frac{1}{4} \Delta \mathcal{A} = \frac{2\pi}{A^2 |G'(\xi_4)| (\xi_3 - \xi_1)}. \quad (22)$$

In the nonextreme case, the action is

$$I_E = -\frac{1}{4}(\Delta \mathcal{A} + \mathcal{A}_{\rm bh}) = \frac{2\pi}{A^2 |G'(\xi_4)| (\xi_3 - \xi_1)}, \quad (23)$$

where we have used the condition $\xi_2 - \xi_1 = \xi_4 - \xi_3$ to cancel the second contribution from $\Delta\mathcal{A}$ with the contribution from $\mathcal{A}_{\rm bh}$.

The limit $r_+ A \ll 1$ may be regarded as a point particle limit, as it represents a black hole small on the scale set by the acceleration. It is in this limit that we would expect to reproduce the result of [9] on the probability for strings to break, forming monopoles at the free ends. In this limit, both the extreme and nonextreme instantons satisfy $r_+ \approx r_-$ (that is, $q \approx m$). The mass per unit length of the string in this limit is

$$\mu \approx r_+ A, \quad (24)$$

and the action (22) and (23) in this limit is

$$I_E \approx \frac{\pi r_+}{A} \approx \frac{\pi m^2}{\mu}, \quad (25)$$

in agreement with the calculation of [9], which found that the action was $I_E = \pi M_m^2/\mu$, where M_m was the monopole mass.

If it is not topologically stable, the string is far more likely to break and form monopoles than it is to break and form black holes, as we do not expect that this semiclassical treatment is appropriate if the black hole mass m is less than the Planck mass, while the monopole mass is typically of the order of $10^{-2} M_{\rm Planck}$. However, even certain kinds of strings that would be topologically stable in flat space can break by the pair creation of black holes [10,11]. Since the mass per unit length μ for realistic cosmic strings is typically of the order $10^{-6} M_{\rm Planck}/l_{\rm Planck}$, breaking to form either monopoles or black holes is extremely rare, and the effect of these tunneling processes on cosmic string dynamics is negligible.

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