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Einstein and Yang-Mills Theories in Hyperbolic Form without Gauge Fixing

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The evolution of physical and gauge degrees of freedom in the Einstein and Yang-Mills theories are separated in a gauge-invariant manner. We show that the equations of motion of these theories can be written in flux-conservative first-order symmetric hyperbolic form where the only nonzero characteristic speed is that of light. This dynamical form is ideal for global analysis, analytic approximation methods such as gauge-invariant perturbation theory, and numerical solution.

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One of the prevailing issues facing general relativity, indeed any gauge theory, is the separation of physical from gauge degrees of freedom. This conceptual difficulty is encountered in the generation of solutions of the field equations, proofs of existence and uniqueness of solutions, and attempts at quantization. In this Letter we present explicitly *hyperbolic* forms [1] of the Einstein and Yang-Mills equations of motion that clearly display the dynamics of these theories *without fixing a gauge*. (The constraint equations remain elliptic.) The basic strategy, which is applicable to any gauge theory, is to take an additional time derivative of the equations of motion, use the constraint equations, and guarantee equivalence to the original theory *via* appropriate choice of Cauchy data. This completes the program to cast general relativity in $3 + 1$ form [2–6] by integrating it with other efforts directed to finding hyperbolic formulations of general relativity [7–10].

Our hyperbolic formulation preserves complete spatial covariance by means of an arbitrary shift vector. The standard $3 + 1$ treatment [3,4] is gauge covariant in this sense but *not* hyperbolic. Our formulation does require a condition on the time slicing to deal with the time-reparametrization invariance of the theory.

A hyperbolic formulation of general relativity is valuable for many applications. The study of analytic approximations can be given a rigorous foundation. Gauge-invariant perturbation theory [11] arises naturally as a perturbative reduction of the new equations [12]. Problems in global analysis, regarding the existence and uniqueness of solutions [13,14], take on a new light

when viewed with the powerful tool of hyperbolic theory [8,15,16]. Insights into quantum gravity and the problem of time seem likely, given an understanding of the precise role of time slicing necessitated by hyperbolicity.

For numerical relativity, the importance of a hyperbolic formulation cannot be overstated. There are many algorithms for solving the hyperbolic equations of fluid dynamics that can now be applied to general relativity. More fundamentally, the isolation of physical from gauge effects means that a numerically generated spacetime can have a closer connection with a desired astrophysical scenario. In the “grand challenge” effort to solve Einstein’s equations for the inspiral and coalescence of compact binaries—a process expected to be observable in gravitational radiation by LIGO and other detectors—one of the difficult problems is the treatment of the horizon of the black hole: gauge degrees of freedom can propagate faster than light and can thus escape from the black hole. In a hyperbolic formulation whose only nonzero speed of propagation is that of light, the horizon again becomes a natural physical boundary. The hyperbolic formulation is also ideal for the treatment of gravitational radiation in numerically generated spacetimes as it makes manifest a split between background and propagating radiation that has long been assumed in approximate calculation schemes [17], in the extraction of gravitational radiation wave forms at finite radius, and in the imposition of outgoing wave boundary conditions [18].

We first demonstrate the procedure in the simpler context of Yang-Mills field theory in flat spacetime

(cf. [16]). The Yang-Mills field strength $F^a_{\mu\nu}$ is given in terms of the vector gauge potential A^a_μ by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu, \quad (1)$$

where ∂_μ indicates ordinary partial differentiation in the x^μ direction in a Minkowski coordinate frame and f^a_{bc} are the structure constants of the Yang-Mills gauge group G . With D_μ indicating a gauge covariant derivative, the Yang-Mills field equations in the absence of sources consist of three equations of motion (for each gauge index value)

$$D^0 F^a_{i0} + D^j F^a_{ij} = 0 \quad (2)$$

and a constraint

$$D^j F^a_{j0} = 0. \quad (3)$$

The Bianchi identity is

$$0 = D_\lambda F^a_{\mu\nu} + D_\mu F^a_{\nu\lambda} + D_\nu F^a_{\lambda\mu}. \quad (4)$$

This is identically satisfied given the definition of $F^a_{\mu\nu}$ and does not need to be imposed separately.

A hyperbolic wave equation for F^a_{i0} is obtained by taking a covariant time derivative of (2) and adding a spatial gradient of (3):

$$D^\mu D^0 F^a_{i0} + D_0 D^j F^a_{ij} + D_i D^j F^a_{j0} = 0. \quad (5)$$

Interchanging the order of covariant differentiations produces (gauge) curvature terms, which may be combined using the antisymmetry of the structure constants. A covariant divergence of the Bianchi identity is then used to give the nonlinear wave equation

$$D^\mu D_\mu F^a_{i0} + 2f^a_{bc} F^b_{ij} F^c_{j0} = 0. \quad (6)$$

The full second-order system of equations consists of the wave equation (6), the constraint (3), and the definition of $F^a_{\mu\nu}$ in (1). [Combining the definition of F^a_{i0} in (1) with the wave equation (6) would produce a third-order hyperbolic equation, with principal part $\square\partial/\partial t$, for $A^a_{i\cdot}$] This system is hyperbolic with elliptic conditions for initial data, and its solution is unique once Cauchy data have been specified on an initial spacelike hypersurface. The Cauchy data consist of an arbitrary gauge potential A^a_0 , the pair A^a_i and F^a_{i0} consistent with the constraint (3), and $D^0 F^a_{i0}$ such that the Yang-Mills equation of motion (2) holds on the initial slice. With these data, one can prove the hyperbolic system is equivalent to the original Yang-Mills equations, yet no gauge-fixing condition has been imposed.

The equations of motion (6) and the definition of F^a_{i0} can be put in flux-conservative first-order symmetric hyperbolic form. The magnetic part of the equations, implicit in the Bianchi identity, must now be used explicitly. Introducing the derivatives of the field strength $G_\lambda^a{}_{\mu\nu} = D_\lambda F^a_{\mu\nu}$ as new variables, one finds

$$D^0 G^a{}_{\mu\nu} + D^k G_k{}^a{}_{\mu\nu} = -2f^a_{bc} F^b{}^\lambda{}_\mu F^c{}_{\lambda\nu}, \quad (7)$$

$$D^0 G_k{}^a{}_{\mu\nu} - D_k G^{0a}{}_{\mu\nu} = -f^a_{bc} F^b{}^0{}_k F^c{}_{\mu\nu}. \quad (8)$$

The unknowns of the first-order hyperbolic system are A^a_i , $F^a_{\mu\nu}$, and $G_\lambda^a{}_{\mu\nu}$, and the equations consist of the definitions of F^a_{i0} and $G^a{}_{\mu\nu}$, (7) and (8).

From the first-order form, one sees that A^a_j and $F^a_{\mu\nu}$ propagate with speed zero: that is, they are simply "dragged along" the time axis during the evolution. It is only the derivatives of the field strength that propagate with the speed of light ($c = 1$). It must be emphasized that A^a_0 is *not* an unknown of the first-order system and that only the fields that propagate with nonzero speed are gauge covariant. A gauge must be chosen to specify A^a_j , but no gauge-fixing condition is required for hyperbolicity.

A hyperbolic formulation for general relativity can be found by a similar procedure [1] (cf. [9] where complete spatial gauge covariance is not present because of the choice of a zero shift vector). Consider a globally hyperbolic manifold of topology $\Sigma \times R$ with the metric

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (9)$$

where N is the lapse and β^i is the shift. Introduce the noncoordinate coframe,

$$\theta^0 = dt, \quad \theta^i = dx^i + \beta^i dt, \quad (10)$$

with corresponding dual (convective) derivatives

$$\partial_0 = \partial/\partial t - \beta^i \partial/\partial x^i, \quad \partial_i = \partial/\partial x^i. \quad (11)$$

Note that $[\partial_0, \partial_i] = (\partial_i \beta^k) \partial_k = C_{0i}{}^k \partial_k$, where the C 's are the structure functions of the coframe, $d\theta^\alpha = -\frac{1}{2} C_{\beta\gamma}{}^\alpha \theta^\beta \wedge \theta^\gamma$.

The natural time derivative for evolution is [4]

$$\hat{\partial}_0 = \partial_0 + \beta^k \partial_k - \mathcal{L}_\beta = \partial/\partial t - \mathcal{L}_\beta, \quad (12)$$

where \mathcal{L}_β is the Lie derivative along the shift vector in a time slice Σ . In combination with the lapse as $N^{-1} \hat{\partial}_0$, this is the derivative with respect to proper time along the normal to Σ , and it always lies inside the light cone, in contrast to $\partial/\partial t$. It has the useful property that $[\hat{\partial}_0, \partial_i] = 0$. The extrinsic curvature K_{ij} of Σ is given by

$$\hat{\partial}_0 g_{ij} = -2NK_{ij}. \quad (13)$$

One employs a procedure parallel to that used in Yang-Mills theory. The spatial metric g_{ij} is analogous to A^a_i , the shift β^k to A^a_0 , and the extrinsic curvature K_{ij} of Σ to F^a_{i0} . The lapse N is a new feature present in time-reparametrization invariant theories.

In four dimensions, Einstein's theory, $R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda{}_\lambda)$, leads to six equations of motion from R_{ij} , three "momentum constraints" from R_{0i} , and the Hamiltonian constraint from $G^0_0 = \frac{1}{2}(R^0_0 - R^k{}_k)$. The hyperbolic form of Einstein's theory is obtained by taking a time derivative of the equations of motion and subtracting spatial gradients of the momentum constraints,

$$\hat{\partial}_0 R_{ij} - \bar{\nabla}_i R_{0j} - \bar{\nabla}_j R_{i0} = \Omega_{ij}. \quad (14)$$

(Barred quantities are defined in the hypersurface Σ .)

Expressing (14) in a 3 + 1 decomposition, one finds

$$\Omega_{ij} = N \square K_{ij} + J_{ij} + S_{ij}, \quad (15)$$

where $\square = -N^{-1} \hat{\partial}_0 N^{-1} \hat{\partial}_0 + \bar{\nabla}^k \bar{\nabla}_k$ is the physical wave operator for arbitrary β^k . If we denote the trace of the extrinsic curvature by $H = K^k{}_k$, then

$$\begin{aligned}
J_{ij} = & \hat{\partial}_0(HK_{ij} - 2K_i^k K_{jk}) + (N^{-2}\hat{\partial}_0 N + H)\bar{\nabla}_i\bar{\nabla}_j N - 2N^{-1}(\bar{\nabla}_k N)\bar{\nabla}_i(NK_j^k) + 3(\bar{\nabla}^k N)\bar{\nabla}_k K_{ij} \\
& + N^{-1}K_{ij}\bar{\nabla}^k(N\bar{\nabla}_k N) - 2\bar{\nabla}_i(K_j^k\bar{\nabla}_k N) + N^{-1}H\bar{\nabla}_i\bar{\nabla}_j N^2 + 2N^{-1}(\bar{\nabla}_i H)(\bar{\nabla}_j N^2) - 2NK^k_{(i}\bar{R}_{j)k} - 2N\bar{R}_{kijm}K^{km}
\end{aligned} \tag{16}$$

[where $M_{(ij)} = \frac{1}{2}(M_{ij} + M_{ji})$] and

$$S_{ij} = -N^{-1}\bar{\nabla}_i\bar{\nabla}_j(\hat{\partial}_0 N + N^2 H). \tag{17}$$

For Ω_{ij} to produce a wave equation, S_{ij} must be equal to a functional involving fewer than second derivatives of K_{ij} . This can apparently be accomplished in a number of ways and constitutes the imposition of a slicing condition on the spacetime. It is necessary to show that the slicing condition can be imposed without spoiling the hyperbolic nature of the evolution system.

A clear and simple slicing condition is the harmonic condition (cf. [9] when $\beta^k = 0$)

$$\hat{\partial}_0 N + N^2 H = 0. \tag{18}$$

[This can easily be generalized by adding an ordinary well behaved function $f(t, x)$ to the right-hand side.] Imposing (18) for all time amounts to imposing an equation of motion for N . The complete system of equations of motion is now the wave equation (15) for K_{ij} , the harmonic slicing condition (18), and the definition (13) of the extrinsic curvature. The Cauchy data for the full system, to be given on an initial slice Σ , are g_{ij} and K_{ij} consistent with the Hamiltonian and momentum constraints, the lapse N , and $\hat{\partial}_0 K_{ij}$, such that the Einstein equations of motion hold on the initial slice. Using the doubly contracted Bianchi identity, one can prove [1] that, with these initial data, the hyperbolic system is fully equivalent to Einstein's theory.

Another useful class of slicing conditions arises from choosing H to be a known function of spacetime $h(t, x)$. In this case, the lapse function N is determined by solution of the time-dependent elliptic problem,

$$\hat{\partial}_0 h = -\bar{\nabla}^k\bar{\nabla}_k N + N(\bar{R} + H^2 - g^{ij}R_{ij}). \tag{19}$$

In this scheme, the Cauchy data are simply g_{ij} and K_{ij} satisfying the constraints and $\hat{\partial}_0 K_{ij}$ from the usual evolution equation. The g_{ij}, K_{ij} system is still hyperbolic,

but the full set of equations is now mixed hyperbolic elliptic. The shift vector can still be specified arbitrarily. Proof of a unique solution proceeds by an iterative method, and equivalence with the usual form of Einstein's equations again employs the twice-contracted Bianchi identity [1].

The harmonic condition is consistent with the natural slicings of stationary spacetimes. For example, suppose one has a spacetime with a timelike Killing vector and a spacelike Killing vector proportional to the shift vector: $\beta^i = f\xi^i$. In this case the evolution of the 3-metric gives $\hat{\partial}_0 g_{ij} = -2\xi_{(i}\partial_{j)}f$ so $H = N^{-1}\xi^l\partial_l f$. Eliminating H with the harmonic slicing condition yields $\xi^l\partial_l(f/N) = 0$, which is clearly true for the Kerr geometry in Boyer-Lindquist coordinates. It is possible and useful in perturbation theory to have the advantages of a specified H and the harmonic slicing condition by choosing the shift vector suitably. However, choosing a shift vector in this particular way seems undesirable for numerical solution of the full field equations because spacetimes evolved in this fashion will tend to develop coordinate singularities as in the stationary spacetimes mentioned above.

In the vacuum case, if we introduce, besides N , g_{ij} and K_{ij} , new variables $a_i = N^{-1}\bar{\nabla}_i N$ —the acceleration of the local Eulerian observers (those at rest in the time slices)—its derivatives $a_{0i} = N^{-1}\hat{\partial}_0 a_i$ and $a_{ji} = \bar{\nabla}_j a_i = a_{ij}$, as well as the derivatives of the extrinsic curvature

$$\hat{\partial}_0 K_{ij} = NL_{ij} \tag{20}$$

and $M_{kij} = \bar{\nabla}_k K_{ij}$, one can cast Eqs. (13), (15), and (18) into complete flux-conservative first-order symmetric hyperbolic form [1]. The unknowns of the first-order system are g_{ij} , N , K_{ij} , L_{ij} , M_{kij} , a_i , a_{ji} , and a_{0i} , and the equations of the first-order system are (13), (18), (20), the definition of a_{0i} , and

$$\hat{\partial}_0 L_{ij} - N\bar{\nabla}^k M_{kij} = J_{ij}, \tag{21}$$

$$\begin{aligned}
\hat{\partial}_0 M_{kij} - N\bar{\nabla}_k L_{ij} = & N[a_k L_{ij} + 2M_{k(i}{}^m K_{j)m} + 2K_{m(i} M_{j)k}{}^m - 2K_{m(i} M^m{}_{j)k} \\
& + 2K_{m(i} (K^m{}_{j)} a_k + a_j) K^m{}_k - a^m K_{j)k}],
\end{aligned} \tag{22}$$

$$\hat{\partial}_0 a_i = Na_{0i}, \tag{23}$$

$$\hat{\partial}_0 a_{ji} - N\bar{\nabla}_j a_{0i} = Na_k [2M_{(ij)}{}^k - M^k{}_{ij} + 2a_{(i} K_{j)k} - a^k K_{ij}] + Na_j a_{0i}, \tag{24}$$

$$\hat{\partial}_0 a_{0i} - N\bar{\nabla}^k a_{ki} = N[-\bar{R}^k{}_i a_k + a_i(H^2 - 2K_{kl}K^{kl} + 2a^k a_k + 2a^k{}_k) + 2a_k a^k{}_i + HM_{ik}{}^k - 2K^{kl}M_{ikl}], \tag{25}$$

where J_{ij} can be found from (16). Notice that the shift is *not* an unknown of the first-order system. The form of the first-order system is independent of the choice of β^k , though it must be specified for solutions. To complete the reduction to first-order form, the three-dimensional Riemann curvature appearing in J_{ij} is expressed in terms of the three-dimensional Ricci curvature using

$$\bar{R}_{mijk} = 2g_{m[j}\bar{R}_{k]i} + 2g_{i[k}\bar{R}_{j]m} + \bar{R}_{g_m[k}g_{j]i}, \quad (26)$$

which in turn is eliminated by substituting

$$\bar{R}_{ij} = R_{ij} + L_{ij} - HK_{ij} + 2K_{ik}K^k_j + a_i a_j + a_{ji}. \quad (27)$$

The four-dimensional Ricci curvature is then eliminated using the Einstein equations.

Note that in spatial dimensions greater than three, the expression for \bar{R}_{mijk} involves the Weyl tensor, which cannot be eliminated using the Einstein equations. The reduction to first-order form in these variables is thus blocked.

One sees that g_{ij} , K_{ij} , N , and a_i all propagate with zero speed with respect to the Eulerian observers: they are dragged along the normal to the foliation by the evolution. Only the derivatives of the extrinsic curvature and the derivatives of the acceleration propagate with the speed of light. These represent time-dependent tidal forces and can be used to form the components of the spacetime Riemann tensor. The only propagating degrees of freedom then are curvatures, as one would expect physically. Equivalently, in the second-order system, the wave equation for K_{ij} can be viewed as defining the notion of radiation as distinctly as possible in the context of a nonlinear, curved space field theory. The inherent separation of the evolution of the spatial metric and extrinsic curvature is a natural starting point for a formal expansion scheme that is gauge invariant at each order.

When linearized around a static background 3-metric, the evolution equations for g_{ij} and the wave equation for K_{ij} decouple. For instance, with a flat space background, the simple wave equation $\square K_{ij} = 0$ for the extrinsic curvature is obtained (assuming harmonic slicing). The evolution equation for the 3-metric contains no new information about the dynamical degrees of freedom. Similarly, one can linearize about static or stationary black hole backgrounds, for instance, the Schwarzschild or Kerr geometries. In the Schwarzschild case, (15) reduces directly to scalar wave equations for the even- and odd-parity radiation modes. (The 3-metric evolution equation is again irrelevant.) Taking an additional time derivative of the scalar equations, one recovers the standard results of gauge-invariant perturbation theory [11]. The pair of scalar wave equations obtained for each (ℓ, m) multipole combination can be matched directly onto a numerically generated interior solution and provides both a gauge-invariant radiation extraction method and a clean prescription for outer boundary conditions (including backscatter

of waves). By refining the assumed exterior background spacetime, arbitrary amounts of physical detail can be incorporated by this general method.

The reasoning we have applied in this paper can be applied to general relativity coupled to other fields with well-posed Cauchy problems, as well as to generally covariant and gauge theories in the broad sense. One sees that the procedure of taking time derivatives and adding further variables can be continued to build a “tower” of equations that, provided suitable initial conditions are given, is equivalent to the original theory. Nothing fundamental is gained by going beyond the stage at which gauge-invariant equations of motion are obtained, as in this paper, but we find the equations that propagate the spacetime Riemann tensor components directly aesthetically appealing. By achieving a hyperbolic formulation of a gauge theory without gauge fixing, one has manifestly physical propagation without the encumbrance that comes from having to impose particular gauge conditions. The physical structure of the theory is revealed with the full gauge symmetry preserved.

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