Sheared Flow Stabilization of the $m = 1$ Kink Mode in Z Pinches

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The effect of a sheared axial flow on the $m = 1$ kink instability in Z pinches is studied numerically by reducing the linearized magnetohydrodynamic equations to a one-dimensional displacement equation. An equilibrium is used that is made marginally stable against the $m = 0$ sausage mode by tailoring its pressure profile. The principal result reveals that a sheared axial flow stabilizes the kink mode when the shear exceeds a threshold that is dependent on the location of the conducting wall. For the equilibria studied here the maximum threshold shear (v_z'/kV_A^0) was about 0.1.

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The $m = 1$ kink instability is well known, both theoretically and experimentally, to plague Z-pinch plasmas [1,2]. (In this paper we refer to a Z pinch as having only axial plasma current.) Certain notable exceptions occur, mainly in Z pinches with a flow velocity, where the flow state is observed to be stable [3,4]. We have been led to reexamine the effects of flow on stability because of the profound implications a stable, high-density Z pinch would have for magnetic confinement thermonuclear fusion [5].

By tailoring the pressure profile of a static Z-pinch equilibrium the $m = 0$ sausage mode can be stabilized, but the equilibrium remains unstable to the internal kink mode [6,7]. The introduction of an axial magnetic field can stabilize the kink mode, but the field also limits the axial plasma current (the Kruskal-Shafranov limit) [8,9]. The pinch must then compress the magnetic field as well as the plasma. An approach that allows high plasma density would be to stabilize the Z pinch without limiting the plasma current.

The axial variation of the kink mode suggests that a constant axial flow may eliminate the variation and stabilize the mode. An infinitely long Z-pinch plasma with a uniform axial velocity would kink at the same rate as a static plasma since the kink mode would simply move at the plasma velocity. However, if the plasma velocity is nonuniform (sheared flow), the kink mode is forced to be axially coherent and is found to be stabilized for sufficient velocity shear. Since the kink mode can move at the average plasma velocity, it is clear that the velocity shear, and not the average velocity, affects the stability. While a uniform flow may stabilize the kink mode in a Z pinch of finite length by convecting the instability to the end of the pinch before it can grow significantly, this would require a super-Alfvénic flow velocity and will not be considered here.

The equilibrium equation for the Z pinch is the radial force balance, which is

$$\frac{B_\theta}{\mu_0 r}\frac{\partial(rB_\theta)}{\partial r} + \frac{\partial p}{\partial r} = 0, \tag{1}$$

where B_θ is the azimuthal magnetic field and p is the plasma pressure. The presence of an axial flow does not affect the equilibrium equation. Performing a functional minimization (or energy principle), an equilibrium condition can be determined that stabilizes the $m = 0$ sausage mode in the static Z pinch [6]:

$$-\frac{d\ln p}{d\ln r} \le \frac{4\Gamma}{2 + \Gamma\beta}, \tag{2}$$

where Γ is the ratio of specific heats and $\beta = 2\mu_0 p/B^2$ is a local measure of the ratio of plasma pressure to magnetic pressure. This condition must be satisfied everywhere in the plasma for stability against the $m = 0$ mode.

The lack of axisymmetry of the kink mode prevents the stability determination by analytical methods, like those used for the sausage mode, and numerical techniques must be used. The ideal MHD (magnetohydrodynamic) equations are linearized. The displacement is assumed to have the form $\xi(r,\theta,z,t) = \xi(r)\exp(\gamma t + im\theta - ikz)$. Here γ is the eigenvalue, and it is, in general, mixed complex [having an oscillatory component $\Im(\gamma)$ and a growing component $\Re(\gamma)$]; m and k are the azimuthal mode number and axial wave number for the eigenmode of interest. For finite length pinches, the axial wave number is limited by the inverse of the pinch length. The linearized MHD equations are combined to yield a pair of first order eigenvalue differential equations for the perturbed total pressure p^* and the radial displacement ξ_r:

$$Xr\frac{\partial p^*}{\partial r} + C_{11}p^* + C_{12}(r\xi_r) = 0, \tag{3}$$

$$Xr\frac{\partial(r\xi_r)}{\partial r} + C_{21}p^* + C_{22}(r\xi_r) = 0, \tag{4}$$

where

$$p^* = \frac{\alpha}{\gamma}\left(p_1 + \frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0}\right) = 2\frac{B_\theta^2}{\mu_0}\frac{\xi_r}{r} - \frac{\rho\alpha^2}{Y}\nabla\cdot\boldsymbol{\xi}, \tag{5}$$

$$C_{11} = i\frac{krX}{\alpha}v_z' + 2\left(\frac{F^2}{\mu_0} + Y\frac{B_\theta^2}{\mu_0}\right), \tag{6}$$

$$C_{12} = X^2 - 2X\frac{B_\theta}{\mu_0}\frac{\partial(B_\theta/r)}{\partial r} - \frac{4B_\theta^2}{\mu_0 r^2}\left(\frac{F^2}{\mu_0} + Y\frac{B_\theta^2}{\mu_0}\right), \tag{7}$$

$$C_{21} = m^2 + k^2r^2 + r^2Y, \tag{8}$$

$$C_{22} = i\frac{krX}{\alpha}v_z' - 2\left(\frac{F^2}{\mu_0} + Y\frac{B_\theta^2}{\mu_0}\right), \tag{9}$$

$$Y = \frac{(\rho\alpha^2)^2}{X\Gamma p + \rho\alpha^2 B_\theta^2/\mu_0}, \tag{10}$$

$$X = \rho\alpha^2 + \frac{F^2}{\mu_0} \tag{11}$$

$$F = \frac{m}{r}B_\theta, \tag{12}$$

$$\alpha = \gamma + ikv_z, \tag{13}$$

$$v_z' \equiv \frac{\partial v_z}{\partial r}. \tag{14}$$

A second order eigenvalue differential equation for the radial displacement is obtained by combining Eqs. (3) and (4).

$$\frac{\partial^2\xi_r}{\partial r^2} + S\frac{\partial\xi_r}{\partial r} + Q\xi_r = 0, \tag{15}$$

where S and Q are functions of the equilibrium, the mode numbers, the eigenvalue, and the radius. The eigenvalue indicates the stability or instability (and oscillation frequency or growth rate) for a particular eigenmode of an equilibrium. This formulation is similar to previous work for the static Z pinch [7,10]. However, here an equilibrium axial flow $v_z(r)$ has been included that modifies the equations.

Since the plasma is surrounded by a rigid wall, the displacement must vanish there, $\xi_r = 0$. At the axis the eigenvalue equation becomes singular. A boundary condition can be determined at the axis by expanding Eq. (15) in a power series about the axis to find

$$\xi_r \propto r^{m-1} \tag{16}$$

at the axis. For the kink mode $m = 1$ this gives a Neumann boundary condition

$$\frac{\partial\xi_r}{\partial r} = 0. \tag{17}$$

The boundary value problem defined by Eq. (15) and the two boundary conditions is solved using a shooting method. A guess is made for the eigenvalue and the differential equation is integrated out to the wall. The eigenvalues for states that do not exhibit overstability behave like those of the Sturm-Liouville problem, and a simple corrector scheme can be used [11]. If the solution overshoots the boundary value, the eigenvalue is lowered, and vice versa.

The equilibria that we studied in this paper are diffuse pure Z pinches, which means that the plasma extends to a rigid conducting wall (no vacuum interface) that is located at r_{wall} and that the plasma current is axial (no axial magnetic field). The equilibria studied are marginally stable to the $m = 0$ mode as defined by the equality of Eq. (2). In parametric form, the equilibria satisfy

$$r = a\frac{(4/5 + \beta)^{1/4}}{\beta^{3/4}} \tag{18}$$

and

$$p = p^0\left(\frac{\beta}{4/5 + \beta}\right)^{5/2}, \tag{19}$$

where a is the characteristic pinch radius and p^0 is the plasma pressure on axis. The magnetic field is calculated from the force balance. The plasma temperature is assumed to be constant, so the shape of the density and pressure profiles are identical. The equilibria have an axial flow that is zero on the axis and has a constant shear,

$$\kappa = \frac{v_z'}{kV_A^0}, \tag{20}$$

where κ is the normalized flow shear. The flow shear is normalized to kV_A^0, where V_A^0 is the nominal Alfvén speed, which is defined by the maximum magnetic field and the maximum plasma density ($V_A = B_\theta/\sqrt{\mu_0\rho}$).

For static equilibria ($\kappa = 0$), the operators are Hermitian, the stability equations are self-adjoint, and the eigenvalues are not mixed complex. If γ^2 is greater than zero, the mode is unstable and the growth rate is γ. If γ^2 is less than zero, the mode is stable and the oscillation frequency is $\omega = -i\gamma$. The normalized growth rate is plotted in Fig. 1 as a function of wall position. The placement of a close-fitting wall is seen to have a stabilizing effect, but when the wall is moved away to $r_{\mathrm{wall}}/a > 4$, the effect vanishes.

The addition of an equilibrium velocity destroys the Hermiticity property of the stability equations. Therefore, the eigenvalues can be mixed complex [12].

$$\gamma = \begin{cases} \Re(\gamma) + i\omega & \text{if unstable,} \\ i\omega & \text{if stable.} \end{cases} \tag{21}$$

We find that the kink mode is stabilized when the flow shear exceeds a threshold value. The threshold shear is dependent on the wall position. This is shown in Fig. 2. At the threshold shear the kink mode is marginally stable, at $\kappa \geq \kappa_{\mathrm{threshold}}$ gives $\Re(\gamma) = 0$, and at $\kappa < \kappa_{\mathrm{threshold}}$ gives $\Re(\gamma) \neq 0$. As the wall is moved away from the plasma, the amount of flow shear required to

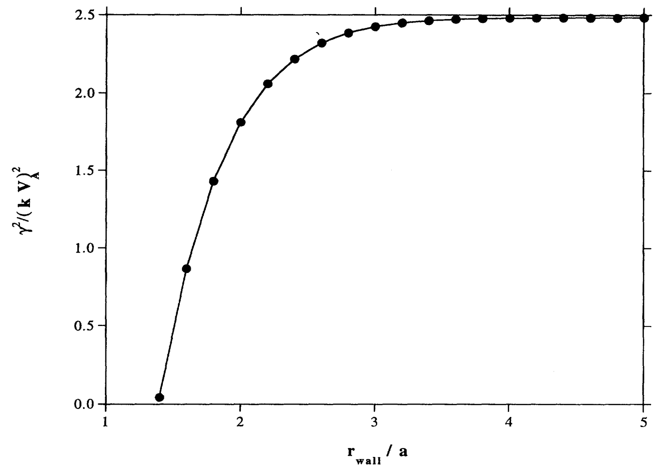

FIG. 1. Wall position effect on the growth rate of the $m = 1$ mode for the static Z pinch. The effect vanishes for $r_{\mathrm{wall}}/a > 4$.

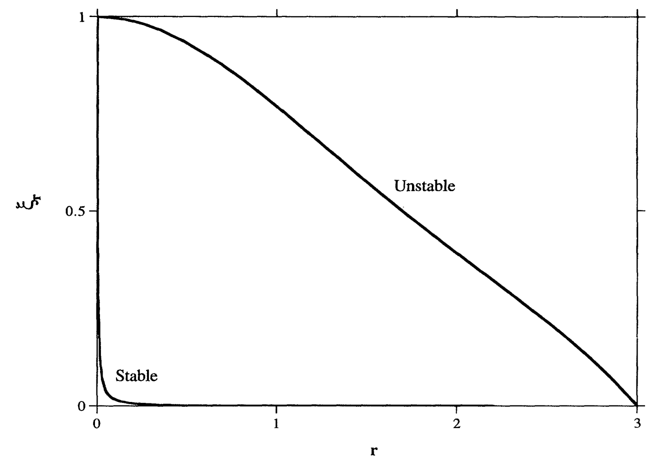

FIG. 3. Eigenmodes of the $m = 1$ mode for a marginally stable ($\kappa = 0.036$) and an unstable (static, $\kappa = 0$) equilibrium with $r_{\mathrm{wall}}/a = 3$. The stable eigenmode is forced to the axis.

stabilize the kink mode increases until $r_{\mathrm{wall}}/a \approx 4$ and is constant beyond that point. This behavior is consistent with the wall stabilization effect for the static Z pinch. The eigenmodes for a marginally stable and an unstable (static) equilibrium are shown in Fig. 3 for $r_{\mathrm{wall}}/a = 3$. The introduction of flow shear localizes the eigenmode at the axis.

We have verified that the flow does not drive the sausage mode unstable. In fact, since the sausage mode has an axial variation like the kink mode, a sheared flow may stabilize the sausage mode, thus relaxing the necessity for equilibrium profile control.

The solutions were found for varying values of the normalized flow shear by discretizing the eigenvalue equation, Eq. (15), using central finite differences on a grid of 1000 cells. The number of grid cells was varied over an order of magnitude and yielded the same results.

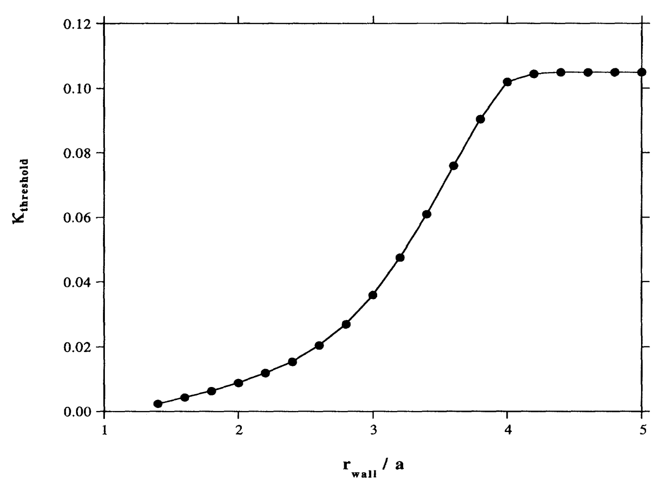

FIG. 2. Threshold shear required to give marginal stability against the $m = 1$ mode as a function of wall position. The effect of the wall position still vanishes for $r_{\mathrm{wall}}/a > 4$.

Our work shows that an equilibrium prescribed by the $m = 0$ marginal stability condition can be stabilized against the $m = 1$ kink mode by the introduction of a sufficiently sheared axial flow. The stabilization effect presented here may be related to the stabilization of ballooning modes in tokamak plasmas with sheared toroidal velocities similar in magnitude to the values presented here [13]. The sheared flow stabilization of the kink mode in Z pinches has important implications for the flow-through Z pinch. A flow-through Z pinch designed with a sheared flow beyond the threshold value could make a simple steady-state fusion device, such as described in Ref. [5].

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