Domain Walls in Wave Patterns

Igor Aranson[1] and Lev Tsimring[2]
[1]*Department of Physics, Bar-Ilan University, Ramat Gan 52900, Israel*
[2]*Institute for Nonlinear Science, University of California at San Diego, La Jolla, California 92093-0402*
(Received 27 July 1994; revised manuscript received 12 July 1995)

We study the interaction of counterpropagating traveling waves in 2D nonequilibrium media described by the complex Swift-Hohenberg equation (CSHE). Direct numerical integration of CSHE reveals novel features of domain walls separating wave systems: wave-vector selection and transverse instability. Analytical treatment is based on a study of coupled complex Ginzburg-Landau equations for counterpropagating waves. At the threshold we find the stationary (yet unstable) solution corresponding to the selected waves. It is shown that sources of traveling waves exhibit long wavelength instability, whereas sinks remain stable. An analogy with the Kelvin-Helmholtz instability is established.

PACS numbers: 05.45.+b, 47.20.–k

Traveling waves (TW) arise naturally in a wide variety of physical settings. In nonequilibrium systems at the threshold of primary instability, only TW with a wave number close to some critical wave number q_c appear. In a large aspect ratio, isotropic system the direction of propagation for TW is not fixed, therefore domains of TW with arbitrary direction of propagation appear [1]. In one-dimensional systems, rotational symmetry degenerates into reflection symmetry, and only left- and/or right-propagating waves remain. Interaction of these counterpropagating waves and the structure of domain walls in 1D systems have been intensively studied in recent years, both theoretically [2–4] and experimentally [5–7]. It has been shown that there are two types of domain walls—passive (sinks) and active (sources). Sources provide a wave-number selection, while sinks do not. On the other hand, sources in 1D exist only at rather small group velocities ($v \propto \varepsilon^{1/2}$, ε is a growth rate of the primary instability), otherwise they do not exist because the trivial state is convectively unstable [3].

The situation is more interesting in the two-dimensional case [1,8,9], where the directions of waves and the orientation of domain walls are not enforced by the boundaries. The question of wave-number selection is therefore transformed into one of *wave-vector* selection, since the domain wall can adjust the direction of incoming or outgoing waves. The domain wall itself may no longer be stationary, but may move in a certain direction if there is no reflection symmetry of wave pattern with respect to the domain wall axis. The second spatial dimension (along the domain wall) opens the possibility for additional instabilities of the wall [10]. In this Letter, we report the analysis of domain walls in two-dimensional systems. We deal here with a particular symmetric case of counterpropagating waves, and domain walls (almost) parallel to the wave vectors, which nevertheless reveals many novel features, including wave-vector selection by sources and their transverse instability. The latter appears to be analogous to the famous Kelvin-Helmholtz instability of a tangential discontinuity of shear flows.

We shall base our study on the complex Swift-Hohenberg equation for the amplitude of the traveling waves in rotationally invariant systems [11]:

$$\psi_t = \varepsilon\psi - \alpha|\psi|^2\psi + i\beta\nabla^2\psi - \gamma(1 + \nabla^2)^2\psi, \quad (1)$$

where $\alpha = 1 - i\tilde{\alpha}$ describes the nonlinear response and β and $\gamma = 1 - i\tilde{\gamma}$ specify the linear dispersion relation of waves. The complex order parameter ψ characterizes the amplitude of waves, the field itself can be written in the form $u(x,y,t) = \psi \exp[-i(\omega_0 - \beta)t] + \text{c.c.}$, where ω_0 is the (unspecified) carrier frequency of linear waves with the critical wave number $q_c = 1$. From (1) it readily follows that the group velocity of waves with $q = 1$ equals $v = 2\beta$. In the following we assume without loss of generality $\beta > 0$, $\varepsilon \ll 1$, and $\beta, v = O(1)$.

Equation (1) for $\tilde{\alpha}, \tilde{\gamma} \ll 1$ has recently been asymptotically derived for the transversely extended laser systems [12,13]. It is also believed that this equation describes traveling wave convection in binary mixtures [14,15], however, in that case it has not been systematically deduced from governing Navier-Stokes equations.

We performed numerical simulations of Eq. (1) with initial conditions corresponding to a domain wall perpendicular to the directions of counterpropagating waves [Fig. 1(a)]. This state is not, however, a stationary solution of (1). As Fig. 1(b) illustrates, near domain wall wave fronts turn at a certain angle to the orientation of the domain wall. This region of selected direction of the wave vector propagates away from the domain wall and tends to occupy the whole region of integration. For small values of $\beta\varepsilon^{-1/2} \lesssim 1$ the selected domain wall remains stable. In the meantime, at large enough $\beta\varepsilon^{-1/2} \gtrsim 1$ a new phenomenon occurs—the selected domain wall is destroyed by a transverse instability [Fig. 1(c)]. This phenomenon is missing in the 1D problem. As the instability enters a strongly nonlinear stage, counterpropagating waves "overturn," scroll, and form a chain of spirals [Fig. 1(d)]. Depending on parameters β and ε the spirals may remain stable (small β) or exhibit a core instability

0031-9007/95/75(18)/3273(4)$06.00

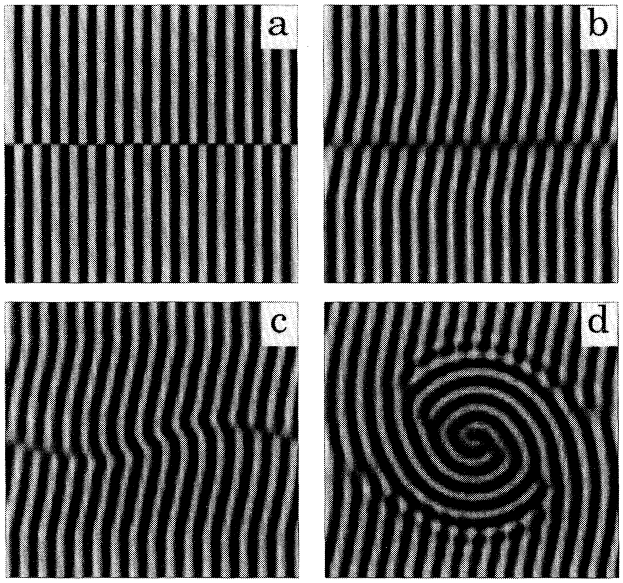

FIG. 1. Sequence of snapshots of numerical simulation of Eq. (1) at $\varepsilon = 0.5$, $\tilde{\alpha} = 0$, $\beta = 1$, $\tilde{\gamma} = 0$ in a box 100×100 with counterpropagating waves separated by a domain wall taken as initial conditions. (a) $T = 0$, (b) $T = 85$, angle selection is seen near the domain wall, (c) $T = 170$, transverse instability of the domain wall, initial stage, (d) $T = 250$, nonlinear stage of the transverse instability, spiral creation.

(larger β) similar to those discovered in the Ref. [16] for the Ginzburg-Landau model.

Let us place a domain wall at the x axis, and assume that wave vectors of counterpropagating waves are almost parallel to the domain wall. We can introduce in (1) an ansatz $\psi = [A_1 \exp(ix) + A_2 \exp(-ix)] \exp[-i\beta t]$. The envelope equations for slowly varying complex amplitudes $A_{1,2}(X, Y, T)$ after appropriate rescaling take a form of coupled Ginzburg-Landau-type equations:

$$\partial_T A_1 + \partial_X A_1 = A_1 + i\partial_Y^2 A_1 - (|A_1|^2 + 2|A_2|^2)A_1 + O(\varepsilon^{1/2}), \quad (2)$$

$$\partial_T A_2 - \partial_X A_2 = A_2 + i\partial_Y^2 A_2 - (|A_2|^2 + 2|A_1|^2)A_2 + O(\varepsilon^{1/2}), \quad (3)$$

where $T = \varepsilon t$, $X = \varepsilon v^{-1} x$, $Y = (\varepsilon/\beta)^{1/2} y$. We assume that $\varepsilon \ll 1$ and therefore keep only lowest-order spatial derivative terms in the amplitude equations. Familiar Newell-Whitehead-Segel-type terms appear at higher orders in ε and have been neglected. We also assumed $\tilde{\alpha} = 0$ as is typical for nonlinear optics [12]. Equations (2) and (3) should be completed by appropriate boundary conditions. For sinks one has to impose boundary conditions corresponding to incoming waves. For sources no-flux conditions can be chosen.

Consider first the wave-vector selection problem. As our numerical simulations of Eq. (1) suggest, a solution with the wave vector exactly parallel to the domain wall, in general, may not be a stationary solution. The domain wall corrects the wave vector in a finite domain, and this domain propagates outwards with some finite velocity. To find the stationary solution of Eqs. (2) and (3), we impose the following boundary conditions:

$$\nabla A_1 = \mathbf{Q}_1 A_1, \quad A_2 = 0 \text{ at } y \to -\infty, \quad (4)$$

$$A_1 = 0, \ \nabla A_2 = \mathbf{Q}_2 A_2, \text{ at } y \to +\infty. \quad (5)$$

Here $\mathbf{Q}_1, \mathbf{Q}_2$ are corrections to the wave vectors to be determined. By virtue of symmetry, $\mathbf{Q}_1 = -\mathbf{Q}_2 = -\mathbf{Q}$. Of course, a one-dimensional domain wall cannot affect the tangential component of the wave vector (here Q_x), so only the Q_y component is selected. In the following we shall assume $Q_x = 0$.

After all the approximations and rescaling we have made, system (2),(3) contains no parameters. Its stationary 1D (X-independent) solutions $S_{1,2}(Y)$ of Eqs. (2) and (3) were found numerically (see Fig. 2, inset) and give a universal value of $Q_{y0} = 0.279\ldots$. The positive value of Q_{y0} (and the corresponding angle between the selected wave vector and the domain wall θ_{so}) indicates that selected waves are emitted from the wall; i.e., this is a source solution. In the original variables the angle θ_{so} selected by the source is equal to $0.279(\varepsilon/\beta)^{1/2}$. This angle seemingly diverges at small $\beta \lesssim \varepsilon$, however, in the latter case previously neglected higher derivatives $-(\varepsilon/\beta^2)\gamma\partial_{YYYY}$ must be added to the right hand sides of (2) and (3). This limits the selected angle at small β (see Fig. 2).

Besides this unique source solution there also is a continuum of sink solutions. For small negative angle θ they still can be described by Eqs. (2) and (3). Although the sink solution with arbitrary small θ can be constructed, only solutions with $\theta < -\theta_{si}$ in fact survive. It is easy to see that the limiting angle for sinks θ_{si} is equal to θ_{so}. Indeed, the domain wall itself selects the waves radiating at the angle θ_{so}. These waves invade the bulk if the magnitude of the y component of the group velocity of

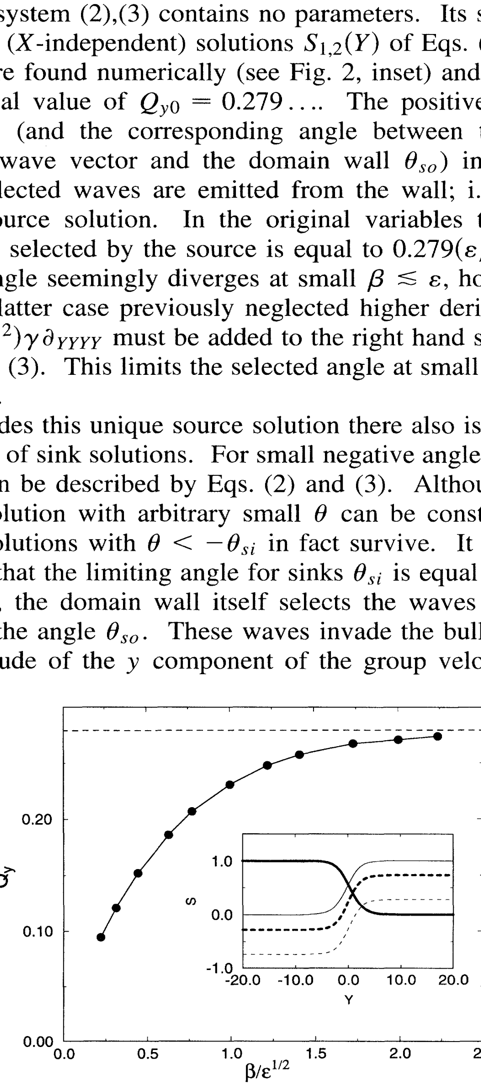

FIG. 2. Selected value of Q_y as a function of $\beta/\sqrt{\varepsilon}$ at $\tilde{\alpha} = 0, \tilde{\gamma} = 0$. The dashed line indicates the asymptotic value $Q_y = 0.279\ldots$. Inset: structure of the stationary domain wall solution of Eqs. (2) and (3) $S_{1,2}$ for $\beta \gg \sqrt{\varepsilon}$; bold solid line, $|S_1|$; bold dashed line, $(\arg S_1)'_Y$; thin solid line, $|S_2|$; thin dashed line, $(\arg S_2)'_Y$.

incoming wave is less than that of the selected wave, or for the dispersion relation of (1), if $|Q_y| < Q_{y0}$. If the opposite inequality holds, the incoming waves do not permit the domain wall to emit the selected wave. Thus we obtain that $\theta_{si} = \theta_{so}$ [17]. If the angle between the wall and the wave vectors is not small, dispersion terms can be neglected in Eqs. (2) and (3), instead the terms $\pm v \sin\theta \partial_Y A_{1,2}$ describing wave propagation in the y direction (towards the domain wall) must be included. 1D coupled amplitude equations of such a form were considered in [3]. In this limit there exists a family of sink solutions for arbitrary Q_y which again do not provide wavenumber selection.

An even wider class of domain wall solutions arises if one relaxes the assumption that waves are exactly counterpropagating. If the angle between wave vectors is not π, one can obtain a stationary moving domain wall solution (see, e.g., [4]). This case will not be considered hereafter.

In the stability analysis of domain walls of traveling waves we assume that parameters $\tilde{\alpha}$ and $\tilde{\gamma}$ are zero. Therefore, there is no bulk Benjamin-Feir instability of traveling waves. We begin with the stability analysis of the source solution, and after that discuss the stability of sinks briefly. We seek the solution of (1) in the form

$$\psi(x,y,t) = \{[S_1(Y) + a_1(X,Y,T)]e^{ix} + [S_2(Y) + a_2(X,Y,T)]e^{-ix}\}e^{-i(\beta\varepsilon^{-1}+Q_y^2)T}, \tag{6}$$

where $S_{1,2}(Y)$ is the stationary domain wall solution of (2) and (3) shown in Fig. 2. Then we obtain one-dimensional linearized equations for $a_{1,2}$:

$$a_{1T} + a_{1X} = ia_{1YY} + (1 - 2|S_1|^2 - 2|S_2|^2)a_1 - S_1^2 a_1^* - 2S_1(S_2 a_2^* + S_2^* a_2), \tag{7}$$

$$a_{2T} - a_{2X} = ia_{2YY} + (1 - 2|S_1|^2 - 2|S_2|^2)a_2 - S_2^2 a_2^* - 2S_2(S_1 a_1^* + S_1^* a_1). \tag{8}$$

Assuming $a_{1,2} = a_{1,2}^r(Y)e^{ikX+\lambda T} + a_{1,2}^l(Y)e^{-ikX+\lambda T}$ (transverse undulations), one arrives at a system of four equations for complex functions $a_{1,2}^r(Y), a_{1,2}^l(Y)$. Solving this system numerically with no-flux boundary conditions for $a_{1,2}^{r,l}$ at $|Y| \to \infty$ yields the dispersion relation $\lambda(k)$. This curve is shown in the Fig. 3. It turns out that the instability is long wave and aperiodic (eigenvalue λ is real). At $k > k_c \approx 0.79$ two imaginary eigenvalues appear which describe propagating disturbances, however, they are not important in the presence of long-wave instability. Note that system (7),(8) does not contain any parameters, so the curve plotted in Fig. 3 gives a full description of the transverse instability of the domain wall for any β and ε, provided that parameters $\varepsilon, \tilde{\alpha}, \tilde{\gamma} \ll 1$.

An analytical description of the transverse instability of domain walls can be achieved in the long-wave limit using phase approximation. The complex amplitudes of two counterpropagating waves can be written in the form $A_{1,2} = |A_{1,2}| \exp i\phi_{1,2}$. For long-wave perturbations

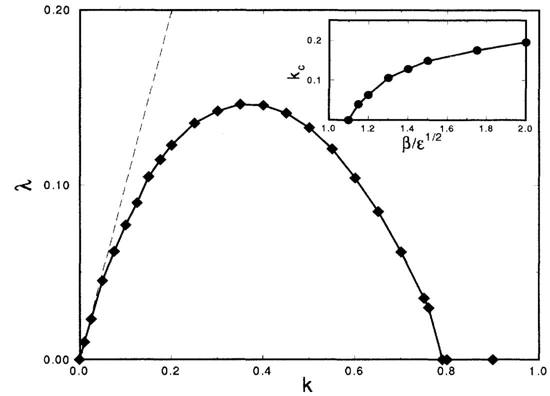

FIG. 3. The growth rate of the transverse instability as a function of its wave number. The dashed line indicates solution of (13) $\lambda = k$. Inset: the upper boundary of the instability band k_c for $\beta \sim \sqrt{\varepsilon}$.

the width of the domain wall is negligibly small, and one can reduce the analysis to phase dynamics in the bulk on both sides from the wall, coupled via boundary conditions across the wall. Far away from the wall the amplitudes of traveling waves $|A_{1,2}|$ follow the evolution of phases $\phi_{1,2}$. Consider a solution in the form $\phi = \mu^{-1}\phi^0 + \phi^1 + O(\mu)$, $|A| = A^0 + \mu A^1 + O(\mu^2)$, and $\tilde{x} = \mu X, \tilde{y} = \mu Y, \tau = \mu T$, where $\mu \ll 1$ is a formal parameter of expansion characterizing the smoothness of solution. Under this assumption one readily obtains in the leading orders equations for phases and amplitudes:

$$\phi_{1,2\tau}^0 \pm 2\phi_{1,2\tilde{x}}^0 = \omega - (\phi_{1,2\tilde{y}}^0)^2, \tag{9}$$

$$A_{1,2}^0 = 1, \quad 2A_{1,2}^1 + \phi_{1,2\tilde{y}\tilde{y}}^0 = 0, \tag{10}$$

where $\omega = Q_y^2$ is a frequency shift of the selected traveling wave solution. Full phases $\phi_{1,2}^0$ can be written as a sum of undisturbed phases of the stationary solution we just found and small perturbations $\phi_{1,2}^0 = \mp Q_y \tilde{y} + \Phi_{1,2}$. Linearized equations for $\Phi_{1,2}$ take the form

$$\Phi_{1,2\tau} \pm \Phi_{1,2\tilde{x}} = \pm 2Q_y \Phi_{1,2\tilde{y}}. \tag{11}$$

Solutions of these equations are

$$\Phi_{1,2} = \xi_{1,2} \exp\left[\tilde{\lambda}\left(\tau \pm \frac{\tilde{y}}{2Q_y}\right)\right] \sin\left[\tilde{k}\left(\tilde{x} + \frac{\tilde{y}}{2Q_y}\right)\right], \tag{12}$$

where $\tilde{\lambda} = \mu^{-1}\lambda$ and $\tilde{k} = \mu^{-1}k$. Now we have to connect ϕ_1 and ϕ_2 using boundary conditions. In order to deduce these conditions one should go beyond the phase equations.

For long-wave perturbations the solution locally is close to a one-dimensional domain wall with its position now being a function of the longitudinal coordinate X. We can define the position of the interface $Y_0(X)$ as given by the condition $|A_1(Y_0(X))| = |A_2(Y_0(X))|$, as for the unperturbed solution. These quasi-one-dimensional solutions have to overlap in some intermediate region

$1 \ll Y \sim \mu^{-1}$ with the solution of the phase equations. Then from (10) we obtain the following boundary condition at the interface in the first order in μ: $\phi_{1\tilde{y}\tilde{y}} = \phi_{2\tilde{y}\tilde{y}}$. The second condition can be obtained using exact symmetries of Eqs. (2) and (3). We notice that these equations can be satisfied by the solution $A_1(x,y,t) = \tilde{A}(x,y,t); A_2(x,y,t) = \tilde{A}(-x,-y,t)$ where $\tilde{A}$ is some function. Looking for the perturbed solution conforming this symmetry, we readily obtain the second boundary condition $\xi_1 = -\xi_2$. The condition $\phi_{1\tilde{y}\tilde{y}} = \phi_{2\tilde{y}\tilde{y}}$ then gives the dispersion relation for the transverse undulations

$$(\lambda + ik)^2 + (\lambda - ik)^2 = 0, \qquad (13)$$

or $\lambda = \pm k$. For the source solution ($Q_y > 0$), the localized mode corresponds to a positive sign, $\lambda = k$. So, we obtained that in the long-wave limit $k \rightarrow 0$ there is an instability of sources with a growth rate proportional to the wave number of disturbances (in dimensional variables $\lambda = 2\beta k$). From Fig. 3 it is obvious that numerically found dependence $\lambda(k)$ indeed agrees with analytical dispersion relation (13) at $k \rightarrow 0$.

The stability of sinks at small θ can be studied using the same framework. Indeed, sinks correspond to $Q_y < 0$, and in order to get a localized solution a negative sign of λ must be chosen. For large negative θ, Eqs. (2) and (3) as well as (9) and (10) do not hold, however, the stability of sinks is guaranteed by the fact that incoming waves trap all the perturbations in the core of the domain wall. Therefore we conclude that sinks are stable with respect to long-wave transverse undulations.

Our stability analysis is not applicable for $\beta \ll 1$, because for $\beta = O(\sqrt{\varepsilon})$ higher order terms in Eqs. (7) and (8) should be taken into account. At small β also $Q_y \rightarrow 0$ and our phase approximation breaks down. Indeed, numerical solution of Eqs. (7) and (8) with added Newell-Whitehead-Segel terms $\sim(\pm i\partial_{\tilde{x}} + \partial_{\tilde{y}}^2)^2 a_{1,2}, \tilde{x} = 2\sqrt{\varepsilon}x, \tilde{y} = \varepsilon^{1/4}y$ shows that the instability band shrinks and disappears for some critical $\beta \approx 1.1\varepsilon^{1/2}$ (see inset in Fig. 3). This is also consistent with our numerical simulations.

It is interesting to note that the transverse instability of domain walls is very similar to the Kelvin-Helmholtz instability of the tangential discontinuity between counterpropagating flows. Instead of flows we deal here with waves which, however, carry energy and momentum. Matching phases of traveling waves across the interface yields the dispersion relation (13), which bears exactly the same form as the dispersion relation for the Kelvin-Helmholtz instability [18]. The physical reason for the instability here is that transverse displacement changes the wave numbers on both sides of the interface, and due to dispersion it creates a group velocity difference, which in turn moves the interface in the direction of the initial displacement.

Our results indicate that although sources play a key role in the pattern selection for 1D systems [2,3,5–8], their one-dimensional analogs in extended 2D systems are of lesser significance since they are typically destroyed by the transverse instability. This conclusion is supported by the early observations of domain walls of counterpropagating waves in the binary mixture convection in 2D cells [8,9]. These "zipper states" were relatively easy to observe in a small aspect ratio cell, whereas in a bigger cell an instability sets in and destroys them [8]. On the other hand, we found that the band of transverse instability collapses at higher values of ε. This agrees qualitatively with observations of zipper states in extended lasers [19].

We are grateful to K. Eaton, A. LaPorta, and C. Surko who attracted our attention to the problem of zipper states in wave patterns and shared their experimental results prior to publication. We benefited from discussions with Q. Feng, J. Moloney, L. Kramer, H. Levine, and M. Rabinovich. L. T. wishes to acknowledge the hospitality of the Bar-Ilan University. I. A. acknowledges the hospitality of the University of Bayreuth. The work of I. A. was supported in part by Rashi Foundation and by Alexander von Humboldt Stiftung. L. T. was supported in part by the U.S. Department of Energy (Grant No. DOE/DE-FG03-95ER14516).

[1] A. LaPorta and C. Surko (to be published).
[2] M. C. Cross, Phys. Rev. A **38**, 3593 (1988).
[3] P. Coullet, T. Frisch, and F. Plaza, Physica (Amsterdam) **62D**, 75 (1993).
[4] B. A. Malomed, Phys. Rev. E **50**, R3310 (1994).
[5] A. Joets and R. Ribotta, J. Stat. Phys. **64**, 981 (1991).
[6] P. Kolodner, Phys. Rev. A **46**, 6452 (1992).
[7] E. Kaplan and V. Steinberg, Phys. Rev. Lett. **71**, 3291 (1993).
[8] C. M. Surko, P. Kolodner, A. Passner, and R. W. Walden, Physica (Amsterdam) **23D**, 220 (1986).
[9] V. Steinbreg, E. Moses, and J. Fineberg, Nucl. Phys. (Proc. Suppl.) **B2**, 109 (1987).
[10] The instability of 1D sources of traveling waves found by Cross [2] is presumably an exponentially weak finite-size effect related to the reflection from the boundaries.
[11] This equation was apparently first introduced by B. A. Malomed, Z. Phys. B **55**, 241 (1984).
[12] J. Lega, J. V. Moloney, and A. C. Newell, Phys. Rev. Lett. **28**, 2978 (1994).
[13] K. Staliunas, Phys. Rev. A **48**, 1573 (1992).
[14] P. C. Hohenberg and M. Cross, Rev. Mod. Phys. **65**, 851 (1993).
[15] M. Bestehorn, R. Friedrich, and H. Haken, Z. Phys. B **75**, 265 (1989).
[16] I. Aranson, L. Kramer, and A. Weber, Phys. Rev. Lett. **72**, 2316 (1994).
[17] This argument is valid only in the case of non-small-group velocity of waves, otherwise more subtle effects of diffusion and dispersion must be taken into account.
[18] H. Lamb, *Hydrodynamics* (Dover, New York, 1945), 6th ed.
[19] Q. Feng (private communication).

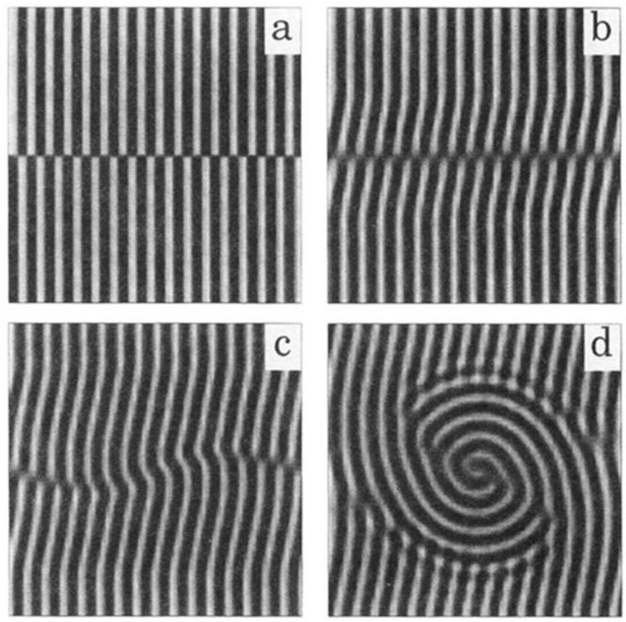

FIG. 1. Sequence of snapshots of numerical simulation of Eq. (1) at $\varepsilon = 0.5$, $\tilde{\alpha} = 0$, $\beta = 1$, $\tilde{\gamma} = 0$ in a box 100×100 with counterpropagating waves separated by a domain wall taken as initial conditions. (a) $T = 0$, (b) $T = 85$, angle selection is seen near the domain wall, (c) $T = 170$, transverse instability of the domain wall, initial stage, (d) $T = 250$, nonlinear stage of the transverse instability, spiral creation.