Dynamical Behavior of a Dissipative Particle in a Periodic Potential Subject to Chaotic Noise: Retrieval of Chaotic Determinism with Broken Parity

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Dynamical behaviors of a dissipative particle in a periodic potential subject to chaotic noise are reported. We discovered a macroscopic symmetry breaking effect of chaotic noise on a dissipative particle in a multistable system emerging, even when the noise has a uniform invariant density with parity symmetry and white Fourier spectrum. The broken parity symmetry of the multistable potential is not necessary for the dynamics with broken symmetry. We explain the mechanism of the symmetry breaking and estimate the average velocity of a particle under chaotic noise in terms of unstable fixed points.

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Success in explaining randomness of physical systems by the deterministic chaos is among the most important progress of recent statistical physics. Tent map chaos is known to have the same randomness as the coin tosses which are universally presumed completely random [1]. Therefore it was assumed that the sequence of tent map chaos is far from its deterministic nature and that there is almost no difference between the chaotic sequence and the similar probabilistic random sequence except for their microscopic structure.

However, recent studies in complex systems indicate that there exists a condition in which an undiscovered order of chaos emerges. "Chaotic itinerancy" [2–4] and "evolution to edge of chaos" [5,6] are some interesting examples. The effect of chaotic dynamics on information processing has also been studied [7–11]. It was also found that the microscopic time correlation of chaos is important in the learning process of chaotic time series by neural networks [12]. One notices that these complex systems are generally multistable. Thus it seemed that the effect of chaotic sequence on these multistable systems differs qualitatively from that of similar probabilistic random sequences. However, the high dimensionality of the multistable systems of neural networks makes the study of the effect of a chaotic sequence difficult. Therefore it was required to select a simple multistable system of low dimension in order to understand clearly the difference between the effects of chaotic sequence and probabilistic random sequence. In this paper, we demonstrate unexpected behavior of a dissipative particle in simple multistable systems subject to chaotic noise and clarify the reason for the peculiar behavior.

Let us consider a particle subject to an external chaotic stimulation in a periodic potential with an additional positive gradient as in Fig. 1. Then a dissipative particle obeys the equation

$$\frac{dx}{dt} = \frac{\partial V}{\partial x} + \sum_{j=-\infty}^{\infty} \xi_j \delta(t - j), \qquad (1)$$

where ξ_j is a chaotic time series. The potential is $V(x) = V_0(x) + ax$, where V_0 is any periodic potential and a is a constant. In this paper, we report the results of our study using a piecewise linear potential as a periodic potential $V_0(x) = h - (h/L)\,|x[\mathrm{mod}\,(2L)] - L|$ for $x \geq 0$, $V_0(-x) = V_0(x)$, where L and h are arbitrary constants, simply because comparison with a theoretical analysis is easy in this case. But it is easily verified that the central result is the same for a smooth periodic potential. In the following we study a discretized equation,

$$x_{n+1} = x_n - \left.\frac{\partial V}{\partial x}\right|_{x=x_n} + \xi_n \qquad (n = 0, 1, 2, \ldots)\,, \qquad (2)$$

which is obtained by integrating Eq. (1) from t_n to $t_{n+1} = t_n + 1$. It can also be verified that a choice of $\Delta t \equiv t_{n+1} - t_n$ does not essentially alter the following results. In the present work, we mainly use chaotic time series produced by a tent map or a Bernoulli shift:

$$\xi_n = -\eta_n\,, \qquad (3)$$

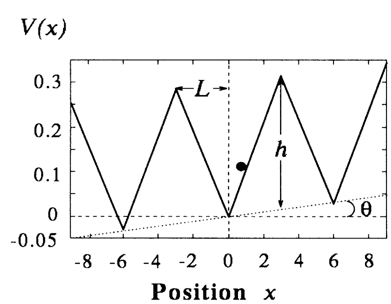

FIG. 1. Periodic potential with constant gradient a (=tan θ), period $2L$, and height h. One particle moves under this potential. In this figure $a = 0.005$, $L = 3.0$, and $h = 0.3$.

$$\eta_{n+1} = f(\eta_n) = \begin{cases} -2|\eta_n| + 1/2 & \text{(tent map)}, \\ 2\eta_n - 1/2\,\mathrm{sgn}(\eta_n) & \text{(Bernoulli shift)}. \end{cases} \quad (4)$$

The minus sign in Eq. (3) is only for a demonstrative purpose.

It is easily verified that the invariant densities of the maps are constant, $\rho(x) = 1$ [for $-0.5 \le x < 0.5$, otherwise $\rho(x) = 0$], which is the same as the uniform random number r_n, $|r_n| < 0.5$, where invariant density $\rho(x)$ is a solution of the Frobenius-Perron integral equation: $\rho(y) = \int dx\, \delta[y - f(x)]\rho(x)$ [13]. Therefore, in the limit of no potential barrier, the dissipative particle under chaotic noise obeys the Brownian motion without a drift term, which is the same with the particle under uniform random noise. In addition, the correlation function of the tent map is δ correlated, which is also identical to uniform random noise. Thus it is useful to characterize the dynamical behavior of a system under chaotic noise by comparing it with that under random noise [14].

One may naively guess that a particle under uniform random noise $[-0.5{:}0.5]$ should move downward (left) on average, since noise is conventionally used in multistable systems to realize a global minimum state, with "annealing" as a typical technique [15,16]. But, surprisingly, we discovered that the particle under chaotic noise produced by a tent map is driven upward (right) against the average potential gradient, which *never occurs* for probabilistic random noise (Fig. 2).

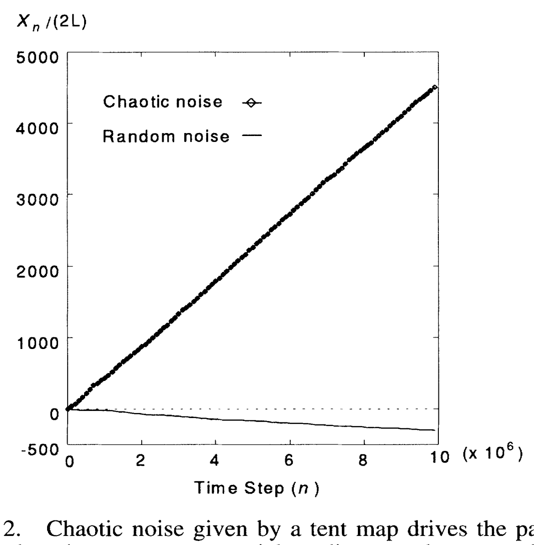

FIG. 2. Chaotic noise given by a tent map drives the particle upward against average potential gradient a, where $a = 0.005$, $L = 3.0$, and $h = 0.3$. On the other hand, random noise drives the particle downward along the average potential gradient. The ordinate shows the distance scaled by the period of the potential. $x_0 = 0$ (initial value). The potential used is the same as in Fig. 1.

Hereafter we discuss the case where parity symmetry of the periodic potential holds, $a = \theta = 0$. Figure 3 shows the numerically obtained transition probability of crossing the potential barrier per unit time as a function of the potential width $2L$ for a fixed potential gradient h/L. The transition probability of a dissipative particle to cross the barrier is much higher under the action of the tent map noise compared to uniform random noise. In the following we show that the hidden deterministic coherence of chaos plays an important role in order to explain the observed phenomena.

One finds that a particle under chaotic noise stays mostly in the neighborhood of a basin of the potential. Thus the particle needs to be forced continuously by the noise having the coherent value to cross the barrier. This condition is satisfied when chaotic noise stays in the neighborhood of an unstable fixed point. The nearer the injected chaotic noise is to the unstable fixed point η^*, the longer η stays in the neighborhood of η^*. By calculating the probability that chaotic noise stays in the neighborhood of the unstable fixed point enough times to drive a particle to cross the barrier one obtains the transition probability of the particle to cross the barrier as a function of the barrier width when the slope of the potential, h/L, is fixed and much smaller than the magnitude of the unstable fixed point:

$$P(L) \sim (1/\Lambda)^{L/|\eta^*|}, \quad (5)$$

where Λ is the slope of the map (2 for tent map and Bernoulli shift) [17]. Numerical simulation for a tent map

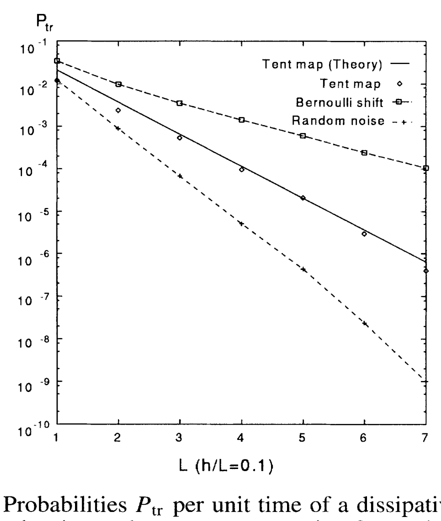

FIG. 3. Probabilities P_{tr} per unit time of a dissipative particle to cross a barrier under a tent map noise for various widths $2L$ of potential barrier with a constant slope $h/L = 0.1$ and no average gradient, $a = 0$, obtained numerically. For comparative purposes, the transition probability in cases of Bernoulli shift and random noise are also shown. All noises have the same invariant densities and amplitudes. We also show a theoretical line given by Eq. (5) for the tent map.

shows that $|\eta^*|$ is 0.4, which should be contrasted with 0.5 as the largest unstable fixed point for this map. The difference of the two values is attributed to the constant slope $h/L(=0.1)$ which functions to decrease the climbing velocity of a particle.

Equation (5) is the general expression first derived for the barrier crossing probability of a particle driven by a chaotic noise, characterized by the local Lyapunov index at the unstable fixed point. It explains why the symmetry break in the dynamics appears in the motion of the particle under a tent map noise. The value of $|\eta^*|$ for positive direction of the particle is $1/2$ and that of the negative direction is $1/6$. Thus the probability of crossing in the positive direction is much higher than the negative direction even when the potential has parity symmetry $V(-x) = V(x)$. The fact that the transition probability in the case of the tent map noise is much higher than in the case of uniform random noise can also be understood by the presence of the coherence of noise in the former case while not in the latter case. Asymmetric motion against the average potential as shown in Fig. 2 is a clear-cut result of the hidden order of the chaotic noise.

A question arises how a dissipative particle in a periodic potential well behaves under action of chaotic noise produced by a map which has symmetric unstable fixed points. Numerical simulation showed that motion of a dissipative particle in the same potential under a sequence of noise created by a Bernoulli shift is not unidirectional but diffusive with a diffusion constant much greater than that under uniform random noise as shown in Fig. 4. The diffusive motion is explained by the equal transition probabilities to cross barriers in the positive and negative directions, as expected from Eq. (5), where the unstable fixed points are $\pm 1/2$ in the Bernoulli shift map. The difference of the diffusion constants between the Bernoulli shift and uniform random noise is again attributed to the presence of short-time correlation in the Bernoulli shift noise as can be seen from the difference of the transition probabilities of crossing the barrier between the cases of the Bernoulli shift and uniform random noise (see Fig. 3).

It should be noted, however, that the transition probability in the case of Bernoulli shift is much greater than the case for the tent map (Fig. 3). Equation (5) for the transition probability is applicable only when the mechanism for a particle to cross the barrier is dominated by a finite time series of monotonically increasing or decreasing chaotic noise. This is the case for the tent map but not for the Bernoulli shift. In the latter case, there is an effective complex mechanism which increases the transition probabilities, in addition to the mechanism mentioned above. In the case of a Bernoulli shift, chaotic noise starting from a neighborhood of a negative unstable fixed point can be reinjected to the neighbor of the same fixed point after one iteration of a small positive value as seen in Fig. 5. This chaotic series at least doubles the effective correlation length and should be contrasted with the case of a tent map, for which the time series must go through a large positive value to be reinjected into the neighborhood of the negative unstable fixed point. Therefore the correlated motion of a particle climbing a slope would largely be interrupted. This observation qualitatively explains why the transition probability in the case of a Bernoulli shift is much larger than the case of a tent map.

Finally, to demonstrate the effect of the chaotic coherence more explicitly, we examined the effect of chaotic noise in the same potential discussed in Fig. 2. Here we used the noise produced by $\eta_{n+1} = f^N(\eta_n)$, where $f(\eta)$ is the tent map [see Eq. (4)] and N is the time of iteration of the tent map. $N = 1$ corresponds to the case discussed above. Figure 6 shows how the speed of the particle driven

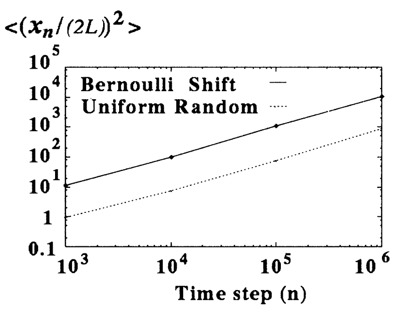

FIG. 4. Under the action of noise generated by the Bernoulli shift for the symmetric potential ($a = 0$), the dynamics is a plain diffusion: $\langle x_n^2 \rangle_{\text{ensemble}} \propto n$. However, the diffusion constant under the Bernoulli shift is much larger than that under uniform random $[-0.5{:}0.5]$, where the two noises have the same invariant densities and amplitudes. The ordinate shows the mean square distance scaled by the period of the potential, $2L$. Data subject to the Bernoulli shift are ensemble averages over η_0 (initial value of the map) of 100 systems, where $L = 2$, $h = 0.2$, $a = 0$, and $x_0 = 0$.

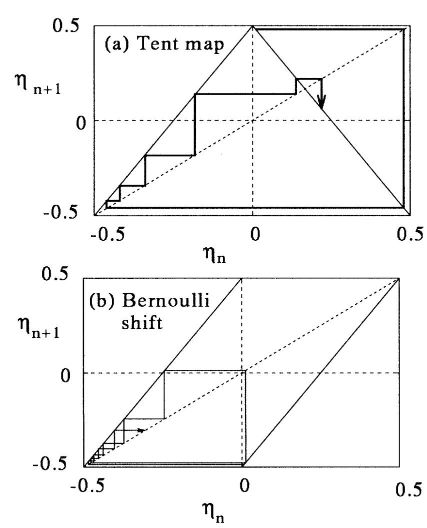

FIG. 5. Typical sequence of (a) the tent map and (b) the Bernoulli shift.

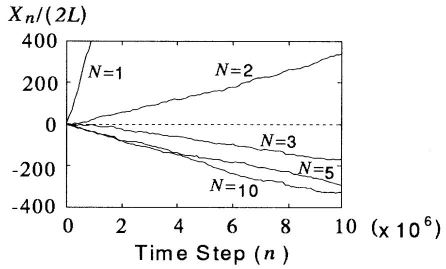

FIG. 6. The trace of a particle motion versus time step. N is the iteration number of the tent map. For $N \to \infty$, the time series is entirely a random number. The potential and the initials used are the same as in Fig. 2.

upward decreases as the number N increases. This phenomenon is also explained by Eq. (5). An increase of N corresponds to an increase of Λ as $\Lambda = 2^N$. Thus an increase of N decreases the transition probability as shown in Eq. (5). Thus systems having a large Lyapunov exponent λ ($\lambda = \ln \Lambda = N \ln 2$ in this case) lose the determinism of a chaotic system; they approach plain random systems. This observation may be related to the recent computer simulation showing that some chaotic systems evolve to $\lambda = 0$ ("edge of chaos"), because chaotic coherence works effectively in systems with a relatively low Lyapunov exponent as demonstrated here. High performance of a Hopfield network with chaotic noise at the edge of chaos [11] may also be explained by the present work.

The drift or diffusion phenomena of chaotic systems without a multistable potential well have been discussed previously [18]. It was reported that the difference of the drift or diffusion rate was caused by the differences of the invariant density or the initial condition of the chaotic noise. We demonstrated here that neither these quantities nor other ordinary statistical quantities, such as correlation function of chaotic noise, explain the difference of the overall dynamics in multistable systems. Even if the higher order statistical quantities are successful in demonstrating different characters of the noise, quantitative prediction or estimation of the dynamics under chaotic noise was found to be difficult by the conventional approach [19,20]. However, the analysis of the effect of chaotic noise on a multistable system by focusing the unstable fixed points of the chaotic noise was found to be a new alternative tool to estimate such characteristics of the dynamics.

In this paper, we emphasized that the emergence of symmetry breaking dynamics under chaotic noise is a special phenomenon in multistable systems. The multistability is widely observed in several fields of nature. Protein motors are good examples. Magnasco showed the condition in which symmetry breaking dynamics (which models the motion of muscle) occurs when a dissipative particle moves in a periodic potential under correlated noise [21]. He proved that the asymmetry of the periodic potential and time correlation of the driving force make it possible to produce broken symmetry dynamics. In contrast to his finding, we reported in this paper another condition for which broken symmetry dynamics emerges. In this condition, asymmetry of the periodic potential is unnecessary if certain chaotic noise works as a driving force. The only condition to produce asymmetric motion is an asymmetric distribution of unstable fixed points of the chaotic noise. The emergence of the broken symmetry dynamics under chaotic noise manifests itself as a typical example of the general findings that "multistability retrieves deterministic nature of chaos."

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