Choptuik Spacetime as an Eigenvalue Problem

Carsten Gundlach

Physics Department, University of Utah, Salt Lake City, Utah 84112 and
Laboratorio de Astrofísica Espacial y Física Fundamental–Instituto Nacional de Tecnología Aerospacial,
*P.O. Box 50727, 28080 Madrid, Spain**

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By fine-tuning generic Cauchy data, critical phenomena have recently been discovered in the black-hole/no-black-hole "phase transition" of various gravitating systems. For the spherisymmetric scalar field system, we find the "critical" spacetime separating the two phases by demanding discrete scale invariance, analyticity, and an additional reflection-type symmetry. The resulting nonlinear hyperbolic boundary value problem, with the rescaling factor Δ as the eigenvalue, is solved numerically by relaxation. We find $\Delta = 3.4439 \pm 0.0004$.

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Recently, Choptuik [1] has studied the gravitational collapse of a real scalar field (massless or massive, minimally or conformally coupled) in spherical symmetry, using an adaptive mesh refinement numerical technique that allows him to study details on very small spacetime scales. To describe his results concisely, we invoke coordinates $\{p, \bar{p}\}$ on the phase space of the spherisymmetric gravitating scalar field, where p is any smooth coordinate such that $p = 0$ is the hypersurface that divides black-hole from no-black-hole spacetimes, while $\bar{p}$ denotes the remaining coordinates. Choptuik's results strongly indicate the following conjectures.

(1) For any choice of coordinate system $\{p, \bar{p}\}$, the mass of sufficiently small black holes is given by $M = f(\bar{p})p^\gamma$ ("scaling"), where $\gamma \sim 0.37$ is a universal exponent.

(2) There is a "critical solution" $\{p = 0, \bar{p} = \bar{p}_*(t)\}$, which acts as an intermediate attractor in a thin sheet surrounding the $p = 0$ hypersurface on both sides ("universality").

(3) This solution shows a discrete homotheticity ("echoing"), or scale invariance, to be defined more precisely below, with a logarithmic rescaling factor $\Delta \sim 3.44$.

More recent research indicates that these properties hold for other self-gravitating systems. Universality, echoing (with $\Delta \sim 0.6$), and scaling (with $\gamma \sim 0.37$) were found in collapse of axisymmetric gravitational waves [2]. Universality, continuous self-similarity, and scaling (with $\gamma \sim 0.36$) were found in perfect fluid collapse with $p = \rho/3$ [3]. The exactly self-similar solution was calculated as an eigenvalue problem [3], and the critical exponent calculated to high precision ($\gamma = 0.355\,8019$) by perturbing it [4]. Critical exponents for other values of k in $p = k\rho$ were calculated in this manner [5] and confirmed in collapse calculations [6], with values strongly dependent on k. Self-similar solutions were also calculated for a complex scalar field [7] and an axion-dilaton combination [8]. A value ($\gamma = 0.387\,106$) of the critical exponent for the complex scalar field was derived by perturbing the self-similar solution [9], but the latter is apparently not an attractor [10].

Universality, scale invariance, and critical exponents indicate an exciting new connection between renormalization group theory and classical general relativity. γ appears to vary from one physical system to another, but its values for vacuum or trace-free matter are remarkably similar in the examples found so far.

In this Letter, we impose echoing, and an additional reflection-type symmetry, in our ansatz, together with analyticity, and solve the resulting nonlinear hyperbolic eigenproblem, instead of evolving and fine-tuning Cauchy data. In the language of renormalization group theory, we find a fixed point of gravitational collapse under a rescaling of space and time by solving the renormalization group equations. In a future paper we intend to calculate γ by perturbing around the fixed point, along the lines of [4,5,9].

The Einstein equations we consider here are

$$G_{ab} = 8\pi G(\phi_{,a}\phi_{,b} - \tfrac{1}{2}g_{ab}\phi_{,c}\phi^{,c}), \qquad (1)$$

in spherical symmetry. The matter equation ${\phi_{,c}}^{;c} = 0$ follows as a Bianchi identity. Following Choptuik, we define the metric as

$$ds^2 = -\alpha(r,t)^2\,dt^2 + a(r,t)^2\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2), \qquad (2)$$

where the remaining gauge freedom is fixed by the condition $\alpha(r = 0, t) \equiv 1$, and auxiliary matter fields as

$$X(r,t) = \sqrt{2\pi G}\,\frac{r}{a}\,\phi_{,r}, \qquad Y(r,t) = \sqrt{2\pi G}\,\frac{r}{\alpha}\,\phi_{,t}. \qquad (3)$$

The symmetry of the attractor observed by Choptuik can be expressed in coordinate language as $Z(r,t) = Z(re^\Delta, te^\Delta)$, where Z stands for any one of α, a, X, and Y, and $\Delta \sim 3.44$ is a constant. Here the zero of t has been adjusted so that $(0,0)$ is the accumulation point of the echos. We introduce auxiliary (nonmetric)

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coordinates where the symmetry appears as a simple periodicity of Z:

$$\tau \equiv \ln(\pm t), \quad \xi \equiv \ln(\pm r/t), \quad Z(\xi,\tau) = Z(\xi,\tau+\Delta). \tag{4}$$

In the following, we use the upper sign ($t > 0$). The equations for the lower sign are obtained by changing the sign of Y in all following equations.

Geometrically, the symmetry can be described as a discrete homotheticity: When we Lie drag g_{ab} along the vector field $\partial/\partial\tau$ by the distance Δ, we obtain $g_{ab}e^{2\Delta}$, while T_{ab} is mapped to T_{ab}. The vector field along which we Lie drag is not unique, however, because we only consider the effect of Lie dragging the finite distance Δ. We parametrize this arbitrariness by introducing a free periodic function $\xi_0(\tau)$ into the coordinate system such that the vector involved in the symmetry is still $\partial/\partial\tau$. At the same time, for clarity of presentation, we absorb Δ into the coordinate τ. We therefore define the coordinates in which we are going to work as

$$\varphi \equiv 2\pi\tau/\Delta, \qquad \zeta \equiv \xi - \xi_0(\tau),$$
$$Z(\zeta,\varphi+2\pi) = Z(\zeta,\varphi). \tag{5}$$

Evans and Coleman [3] found the critical spacetime of spherical fluid collapse by imposing continuous homotheticity. In our coordinates, α, a, and the fluid variables corresponding to X and Y are then functions of ζ alone, the Einstein and matter equations are reduced to a system of nonlinear ordinary differential equations (ODEs), and the solution is uniquely specified by regularity conditions [3,11]. Here we use a similar approach in order to find Choptuik's critical spacetime of scalar field collapse. [After this work was begun, the same approach was used to calculate (continuously) self-similar solutions for the complex scalar field [7] and an axion-dilaton combination [8].]

We solve the field equations for the ζ derivatives of the fields Z as functions of these fields and their φ derivatives. It is convenient to use the new field $g \equiv e^{\xi_0(\varphi)}a/\alpha$ instead of α, and $X_\pm \equiv X \pm Y$ instead of X and Y. The resulting equations are

$$a_{,\zeta} = \tfrac{1}{2}a[(1-a^2) + a^2(X_+^2 + X_-^2)], \tag{6}$$

$$g_{,\zeta} = g(1-a^2), \tag{7}$$

$$X_{+,\zeta} = B_+/(1+D), \tag{8}$$

$$X_{-,\zeta} = B_-/(1-D), \tag{9}$$

where we have introduced the abbreviations

$$z \equiv \left(1 + \frac{2\pi}{\Delta}\frac{d\xi_0}{d\varphi}\right)^{-1}, \qquad D \equiv z^{-1}e^{\zeta}g, \tag{10}$$

$$B_\pm \equiv \frac{1}{2}(1-a^2)X_\pm - a^2X_\mp^2X_\pm - X_\mp \pm z\frac{2\pi}{\Delta}DX_{\pm,\varphi}. \tag{11}$$

There is also one equation containing only the Z and $Z_{,\varphi}$,

$$z\frac{2\pi}{\Delta}\frac{a_{,\varphi}}{a} = \frac{1}{2}[(1-a^2) + a^2(X_+^2 + X_-^2) + a^2D^{-1}(X_+^2 - X_-^2)]. \tag{12}$$

It acts as a constraint, which is conserved by the four "evolution equations" above.

For small enough ζ these equations define a constrained Cauchy problem, with ζ playing the role of time, on the cylinder obtained by identifying φ with period 2π. At $\zeta = -\infty$, corresponding to $r = 0$, we set the boundary conditions $a = 1$ (regularity of the metric) and $\alpha = 1$ (coordinate condition). Expanding the field equations in powers of e^ζ, we find that data obeying these conditions are determined by $\xi_0(\varphi)$ and one more free function $Y_0(\varphi)$, which is defined by the expansion

$$Y(\varphi,\zeta) \equiv Y_0(\varphi)e^{\xi_0(\varphi)}e^{\zeta} + O(e^{3\zeta}). \tag{13}$$

As ζ increases, the Cauchy problem eventually becomes degenerate, when $D = 1$. In analogy to the "sonic point" of the ODEs describing continuously homothetic spacetimes [3,11], we call this line the "sonic line." The equation of radial null geodesics is

$$\frac{d\zeta}{d\varphi} = -\frac{\Delta}{2\pi z}(-1 \pm D^{-1}). \tag{14}$$

The sonic line is therefore the set of points where a null geodesic touches a surface of constant ζ. (In general, it would be a matter characteristic, but for our choice of matter these are identical with the null geodesics.) The solution can be uniquely continued across the sonic line when we impose analyticity. As a technical simplification, we make use of the coordinate freedom in ξ_0 by moving the sonic line to $\zeta = 0$. Then we can enforce analyticity simply by expanding in powers of ζ. We find that regular data near $\zeta = 0$ can be expressed in terms of $\xi_0(\varphi)$ and one more free function $X_{+0}(\varphi)$, which is defined as

$$X_{+0}(\varphi) \equiv X(\zeta=0,\varphi) + Y(\zeta=0,\varphi). \tag{15}$$

We have now formulated a hyperbolic boundary value problem on a rectangle with two sides identified (a finite cylinder) in 1 + 1 dimensions. We have three independent fields, for example, X_+, X_-, and g. On the other hand, there are three free functions in the boundary data, minus one degree of freedom corresponding to translations in φ, plus Δ as the eigenvalue of the problem. By this count we expect solutions to be locally unique, with a discrete spectrum.

We cut the number of degrees of freedom in half by imposing the additional symmetry $Z(\varphi+\pi) = \pm Z(\varphi)$, with the + sign holding for a, g, and ξ_0, and the − sign for X_+ and X_-. It is consistent with the field equations and Choptuik's data [12]. Moreover, Choptuik observed that the massive scalar field has the same attractor as the massless one considered here. The necessary and sufficient condition for this is that ϕ remains bounded, because the mass term in its stress tensor is then dominated by the gradient squared term as ϕ varies on ever smaller spacetime scales. For ϕ to remain bounded its derivatives X and Y must have vanishing zero frequency Fourier components in φ, which in turn requires that all their even frequencies vanish, or else these could be combined to give a zero frequency contribution in the evolution equations. It follows in turn that a, g, and ξ_0 must not have odd frequency components.

As we are dealing with smooth periodic functions, it is useful to decompose all fields into their Fourier components with respect to φ. Integration and differentiation are done in Fourier components. Algebraic operations are done in φ space, which makes our algorithm pseudospectral. Because of the nonlinearity of the problem, dealiasing turns out to be essential for stability. We dealiase convolution sums by using a number of collocation points in φ equal to twice the number of Fourier components.

Because the number of variables is large and the problem is nonlinear in an essential way (there are no regular solutions to the linearized, no gravity, problem), any algorithm is likely to have only a small region of convergence around any solution. We have therefore started with an initial guess sufficiently close to Choptuik's critical spacetime, in order to establish that this solution exists having the echoing symmetry as an exact symmetry, and that it is locally unique, and to calculate it with higher precision than has been possible by fine-tuning Cauchy data. A global search for all solutions that may exist is desirable, but not possible with the present algorithm.

Our manual input into the algorithm is limited to the following guess for Y_0 and Δ: $Y_0 = -2.3\sin\varphi - 0.6\sin3\varphi$, and $\Delta = 3.44$. Here and in the following we fix the translation invariance by defining Y_0 to have no $\cos\varphi$ component. The numbers were estimated from very near-critical collapse data made available by Choptuik [12], after transformation to coordinates (ξ, τ). In a first step, we begin with the very rough guess $\xi_0 = 0$, and shoot from $\zeta = -\infty$ towards increasing ζ. When $D(\varphi, \zeta)$ first gets close to 1 in two points φ, and $X_{-,\zeta}$ is therefore about to become singular in those points, we stop the evolution and calculate a new value of ξ_0 that is designed to "flatten" $D(\varphi)$, i.e., to make it roughly $D(\varphi) \sim 1$ for all φ at that ζ. Then we shoot again, thus iteratively improving ξ_0. After convergence, we read off X_+ at the end point of our one-sided shooting, which by now is close to $\zeta = 0$, and thus have an initial guess for X_{+0} as well.

As an intermediate step, we calculate an initial guess for the values of all fields on a grid in ζ by shooting from both $\zeta = -\infty$ and $\zeta = 0$ to a fitting point, typically $\zeta = -1$. This involves a Taylor expansion around the regular singular point 0 as well as around $-\infty$. Using this expansion and shooting from $\zeta = 0$ transfers the bulk of the error in our improving solution away from the point $\zeta = 0$ to the fitting point $\zeta = -1$, making it easier for the following step to handle.

In the last step, we go over to a standard relaxation algorithm [13]. For the purpose of relaxation the independent variables at each grid point in ζ are the odd Fourier components of X_+ and X_- and the even components of g and ξ_0. a is not considered as independent, but reconstructed at each step from the other fields by solving Eq. (12). Solution of this ODE is by iteration of the corresponding integral equation. (The constant component of a has to be calculated separately.) Between generic grid points in ζ we enforce the discretized ζ derivatives

$$Z_{n+1} - Z_n = hF[(Z_{n+1} + Z_n)/2]. \qquad (16)$$

(The ζ derivative of ξ_0 is zero by definition.) At the boundary $\zeta = -\infty$ we enforce relations between g, X, and Y derived from expanding the field equations, and at $\zeta = 0$ we enforce $D = 1$ and $B_- = 0$. The relaxation part of the algorithm is much simpler than the shooting parts, but the latter appear to have a larger region of convergence, thus serving as a stepping stone.

The boundary data of the solution have been tabulated in Table I. In particular, the echoing period is $\Delta = 3.4439 \pm 0.0004$. The error bars have been obtained combining the results of three different convergence tests. (1) We compare the results obtained for different numbers M of grid points in ζ. As expected the convergence is quadratic, over a wide range of M, but only up to some maximal value. (2) The convergence of the tabulated numbers with increasing number N of Fourier components used in the calculation is rapid ("spectral convergence") for $N \geq 32$. (3) $\zeta = -\infty$ is represented by a finite value of ζ, using a Taylor expansion to one beyond leading order in $\exp\zeta$. As expected, this convergence is quadratic in $\exp\zeta$, over some range of ζ. As long as the difference between runs of different precision has the expected functional form, we can use it to estimate the numerical error. The tabulated data are from a run with $M = 201$ equally spaced points in the interval $-5 \leq \zeta \leq 0$ and $N = 64$ components (half of which vanish) per function, compared with $-6 \leq \zeta \leq 0$, $M = 401$, and $N = 128$, respectively. The three sources of numerical error are comparable for this choice.

We have compared the fields a, α, X, and Y with Choptuik's data, after interpolating to the largest rectangular grid in τ and ξ contained in both data sets, with $-3.2 < \xi < 1.3$. We have evaluated the root mean square of the absolute difference point by point of the fields a, X, and Y (which are bounded and of order one in the solution) and the relative difference in α (which is unbounded above, but bounded below by 1). After adjusting a nonuniversal offset in τ between the data sets, this difference is 3.9×10^{-2} for α, and somewhat smaller for the other fields. By comparison, the estimated root mean square pointwise error in our data is 1.6×10^{-3} in α, and 1.0×10^{-4} or less for the other fields. We have therefore improved the precision with which the Choptuik spacetime is known by 1 to 2 orders of magnitude, while Δ is now known to one part in 10^4. Future improvements are possible.

TABLE I. Decomposition of the boundary data in sines and cosines. Their period is $\Delta = 3.4439 \pm 0.0004$.

Component	ξ_0	Component	Y_0	X_{+0}
const	(1.5813 ± 0.0007)	$\cos\varphi$	0[a]	$(-4.3831 \pm 0.0006) \times 10^{-1}$
$\cos 2\varphi$	$(6.658 \pm 0.006) \times 10^{-2}$	$\sin\varphi$	(-2.364 ± 0.006)	$(-3.2287 \pm 0.0008) \times 10^{-1}$
$\sin 2\varphi$	$(-1.577 \pm 0.002) \times 10^{-1}$	$\cos 3\varphi$	$(-1.46 \pm 0.05) \times 10^{-1}$	$(6.74 \pm 0.05) \times 10^{-3}$
$\cos 4\varphi$	$(-2.014 \pm 0.004) \times 10^{-2}$	$\sin 3\varphi$	$(-9.52 \pm 0.08) \times 10^{-1}$	$(1.017 \pm 0.001) \times 10^{-1}$
$\sin 4\varphi$	$(-3.3 \pm 0.2) \times 10^{-4}$	$\cos 5\varphi$	$(-1.12 \pm 0.05) \times 10^{-1}$	$(2.431 \pm 0.003) \times 10^{-2}$
$\cos 6\varphi$	$(1.979 \pm 0.005) \times 10^{-3}$	$\sin 5\varphi$	$(-4.06 \pm 0.06) \times 10^{-1}$	$(-1.807 \pm 0.008) \times 10^{-2}$
$\sin 6\varphi$	$(2.249 \pm 0.008) \times 10^{-3}$	$\cos 7\varphi$	$(-7.0 \pm 0.4) \times 10^{-2}$	$(-9.85 \pm 0.02) \times 10^{-3}$
$\cos 8\varphi$	$(1.37 \pm 0.01) \times 10^{-4}$	$\sin 7\varphi$	$(-1.73 \pm 0.04) \times 10^{-1}$	$(-2.14 \pm 0.02) \times 10^{-3}$
$\sin 8\varphi$	$(-8.186 \pm 0.004) \times 10^{-4}$	$\cos 9\varphi$	$(-3.9 \pm 0.3) \times 10^{-2}$	$(1.76 \pm 0.01) \times 10^{-3}$
$\cos 10\varphi$	$(-1.886 \pm 0.002) \times 10^{-4}$	$\sin 9\varphi$	$(-7.3 \pm 0.2) \times 10^{-2}$	$(3.116 \pm 0.005) \times 10^{-3}$
$\sin 10\varphi$	$(3.6 \pm 0.1) \times 10^{-5}$	$\cos 11\varphi$	$(-2.1 \pm 0.2) \times 10^{-2}$	$(3.97 \pm 0.02) \times 10^{-4}$
$\cos 12\varphi$	$(3.08 \pm 0.03) \times 10^{-5}$	$\sin 11\varphi$	$(-3.0 \pm 0.1) \times 10^{-2}$	$(-1.195 \pm 0.005) \times 10^{-3}$
$\sin 12\varphi$	$(1.73 \pm 0.02) \times 10^{-5}$	$\cos 13\varphi$	$(-1.1 \pm 0.1) \times 10^{-2}$	$(-4.05 \pm 0.01) \times 10^{-4}$
$\cos 14\varphi$	$(-4.05 \pm 0.09) \times 10^{-6}$	$\sin 13\varphi$	$(-1.24 \pm 0.06) \times 10^{-2}$	$(1.93 \pm 0.02) \times 10^{-4}$
$\sin 14\varphi$	$(-1.371 \pm 0.002) \times 10^{-5}$	$\cos 15\varphi$	$(-5.2 \pm 0.9) \times 10^{-3}$	$(1.53 \pm 0.01) \times 10^{-4}$
$\cos 16\varphi$	$(-3.05 \pm 0.03) \times 10^{-6}$	$\sin 15\varphi$	$(-5.0 \pm 0.2) \times 10^{-3}$	$(5.94 \pm 0.06) \times 10^{-5}$

[a]By definition, to fix translation degree of freedom.

Data files of the solution are available at http://www.laeff.esa.es/~gundlach/.

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*Current address.
Electronic address: gundlach@laeff.esa.es

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