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Reconstructing the Density Operator via Simple Projectors

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We describe the representation of arbitrary density operators in terms of expectation values of simple projection operators. Two representations are presented which yield *nonrecursive* schemes for experimentally determining the density operator of any quantum system. We suggest a possible experimental implementation in quantum optics.

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In the realm of quantum theory a state of a physical system is most generally expressed by its density operator $\hat{\varrho}$. Knowledge of this operator gives complete information of the quantum state. Schemes have been proposed in a number of fields in quantum physics to determine $\hat{\varrho}$ from measurements either explicitly [1–7] or indirectly via quasiprobability distributions [8–11] for mixed states and also for pure states only [12,13].

In this Letter we describe a general method of representing any density operator $\hat{\varrho}$ in terms of expectation values of simple projection operators. Since the expectation values of projectors can, in principle, be determined experimentally, this approach leads to schemes for experimentally determining the density operator.

Our approach differs from previously proposed schemes in quantum optics for determining the density operator in its use of simple projectors which project onto a single or a linear superposition of two basis states [2,5]. We place an emphasis on a "minimalistic" representation which comprises the least number of projection operators and thus leads to the most efficient scheme; it is a generalization of the previous considerations in [2,5].

The plan of the paper is as follows. First we introduce the general idea of our approach, then we cast it into two specific representations and describe their relative virtues. We next describe a quantum optical implementation, and we end with a discussion.

Let us assume that the Hilbert space representing the states of the physical system is of countable dimension N, and let $|m\rangle$ for $m = 1, \ldots, N$ be *any conveniently chosen* orthonormal basis of the space. In cases where the space is infinite in dimension, all expressions containing N here and in the following are infinite also. Our primary aim is to represent the $N^2 - 1$ independent density matrix elements $\varrho_{nm} \equiv \langle n|\hat{\varrho}|m\rangle$ in terms of the expectation values of simple projection operators. Clearly the matrix elements cannot be expressed solely in terms of the $N - 1$ independent expectation values $\varrho_{mm} = \langle |m\rangle\langle m| \rangle$ of the set of the N base state projectors $|m\rangle\langle m|$ because the vital phase information of the coherences, i.e., the complex nature of the off-diagonal elements ϱ_{nm} for $n \neq m$, cannot be derived from the diagonal elements alone.

The simplest possible generalization of the base state projectors is given by the set of projection operators which project onto linear combinations of *two* base states, e.g., onto $c_1|n\rangle + c_2|m\rangle$. The expectation value of such projectors represents the phase information of the coherences in its most elementary form. We show that *one can express $\hat{\varrho}$ in terms of expectation values of such projectors and how to implement it quantum optically.*

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Two representations.—For simplicity let us consider the $\{|n\rangle, |m\rangle\}$ subspace which is spanned by any two basis vectors $|n\rangle$ and $|m\rangle$ for $n \neq m$ and define the state

$$|a\rangle \equiv N_a(|n\rangle + a|m\rangle), \tag{1}$$

where $N_a = 1/\sqrt{1 + |a|^2}$ is a normalization constant and $a \equiv |a|e^{i\alpha}$ is a nonzero coefficient. A corresponding nomenclature is used for a second, different state of the same subspace $|b\rangle = N_b(|n\rangle + b|m\rangle)$, where $b = |b|e^{i\beta} \neq a$. We defer making any further restrictions on the values of a and b, to guarantee independence of the expectation values of the corresponding projectors

$$\hat{A} \equiv |a\rangle\langle a|, \qquad \hat{B} \equiv |b\rangle\langle b| \tag{2}$$

until later.

Let us assume the measurements yielding the expectation values of the projectors $|n\rangle\langle n|, |m\rangle\langle m|, \hat{A}$, and $\hat{B}$ have been performed [14]. The first two expectation values are simply the diagonal elements ϱ_{nn} and ϱ_{mm}. We can combine these expectation values conveniently as

$$\begin{aligned} M_{|a\rangle} &\equiv \mathrm{Tr}\{\hat{\varrho}\hat{A}\} - N_a^2(\varrho_{nn} + |a|^2\varrho_{mm}) \\ &= N_a^2(a\varrho_{nm} + a^*\varrho_{mn}), \end{aligned} \tag{3}$$

where Tr is the trace operation and $M_{|a\rangle}$ stands for the result associated with a measurement of the projector $\hat{A}$. A corresponding expression is obtained for the result $M_{|b\rangle}$ associated with the projector $\hat{B}$. Let us write ϱ_{nm} in terms of its real and imaginary parts $\varrho_{nm} \equiv R + iJ$, and let us define

$$\begin{aligned} m_{|a\rangle} &\equiv \frac{M_{|a\rangle}}{2|a|N_a^2} = R\cos\alpha - J\sin\alpha, \\ m_{|b\rangle} &\equiv \frac{M_{|b\rangle}}{2|b|N_b^2} = R\cos\beta - J\sin\beta. \end{aligned} \tag{4}$$

Solving these equations for R and J yields

$$\begin{aligned} \begin{pmatrix} R \\ J \end{pmatrix} &= \frac{1}{\sin(\beta - \alpha)} \begin{pmatrix} \sin\beta & -\sin\alpha \\ cos\beta & -\cos\alpha \end{pmatrix} \begin{pmatrix} m_{|a\rangle} \\ m_{|b\rangle} \end{pmatrix} \\ &\equiv \boldsymbol{T} \begin{pmatrix} m_{|a\rangle} \\ m_{|b\rangle} \end{pmatrix}. \end{aligned} \tag{5}$$

Clearly this requires $\beta - \alpha \neq k\pi$, where k is any integer. This gives the only restriction on the values of a and b aside from the trivial requirement that $a \neq 0 \neq b$. Applying the outlined procedure to the $\{|n\rangle, |m\rangle\}$ subspaces for $1 \leq n < m \leq N$ allows us to represent $\hat{\varrho}$ in terms of expectation values of $N^2 - 1$ different projectors, due to the condition $\mathrm{Tr}\varrho = 1$. Note that this scheme is intrinsically *nonrecursive*.

We call this the "minimal" representation as it requires this least possible number of projection operators to represent a general density operator and also because it puts almost no restrictions on the states forming the projectors, namely, on the coefficients a and b of Eq. (1).

Though mathematically satisfactory, the minimal representation would be sensitive to experimental errors in a physical implementation. This sensitivity, however, is minimized using *sensitivity optimized states,* i.e., choosing $|a| = |b| = 1$ and $b = \pm ia$ [15]. This sensitivity can be further reduced employing three or more (redundant) states. Let us, for example, look at the case of one more projector state $|c\rangle \equiv N_c(|n\rangle + c|m\rangle)$, where $c = |c|e^{i\gamma}$ in each $\{n, m\}$ subspace. We find that

$$m_{|c\rangle} = \frac{m_{|a\rangle}\sin(\beta - \gamma) - m_{|b\rangle}\sin(\alpha - \gamma)}{\sin(\beta - \alpha)}, \tag{6}$$

where $m_{|c\rangle}$ is given by Eqs. (3) and (4) with $|b\rangle$ replaced with $|c\rangle$. Provided the differences $\alpha - \gamma, \beta - \gamma$, and $\alpha - \beta$ between the phase angles of the states $|a\rangle$, $|b\rangle$, and $|c\rangle$ are not multiples of π, the overparametrization introduced by the extra state can be used to reduce the effect of experimental errors. For example, one could estimate true values of $m_{|a\rangle}$, $m_{|b\rangle}$, and $m_{|c\rangle}$ as the point (x, y, z) on the surface $z(x, y) = [x\sin(\beta - \gamma) - y\sin(\alpha - \gamma)]/\sin(\beta - \alpha)$ which is closest to the point $(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are the experimentally measured values of $m_{|a\rangle}$, $m_{|b\rangle}$, and $m_{|c\rangle}$.

One may still go one step further and consider the particular quadruplet of states

$$\begin{aligned} |a_\pm^{nm}\rangle &\equiv \frac{1}{\sqrt{2}}(|n\rangle \pm |m\rangle), \\ |b_\pm^{nm}\rangle &\equiv \frac{1}{\sqrt{2}}(|n\rangle \pm i|m\rangle), \end{aligned} \tag{7}$$

for $n, m = 1, 2, \ldots, N$. We mention in passing that all such states are normalized except for $n = m$ for which $|a_+^{nn}\rangle \equiv \sqrt{2}|n\rangle$ and $|a_-^{nn}\rangle, |b_\pm^{nn}\rangle \equiv 0$. The set $\{|a_\pm^{nm}\rangle, |b_\pm^{nm}\rangle : m, n = 1, \ldots, N\}$ is an overcomplete basis of the Hilbert space. Let the projection operators [16] which project onto these states be $\hat{A}_\pm^{nm} \equiv |a_\pm^{nm}\rangle\langle a_\pm^{nm}|, \hat{B}_\pm^{nm} \equiv |b_\pm^{nm}\rangle\langle b_\pm^{nm}|$, defined in analogy to Eq. (2). The expectation values of the $2N^2 - N$ different projectors [17] for $n \leq m$ suffice to represent an arbitrary matrix element of $\hat{\varrho}$ as

$$\varrho_{mn} = \mathrm{Tr}\{\hat{\varrho}\,\tfrac{1}{2}[\hat{A}_+^{nm} - \hat{A}_-^{nm} + i(\hat{B}_+^{nm} - \hat{B}_-^{nm})]\}, \tag{8}$$

a form that has already been derived in [2,5]. Now the projectors can be combined to form operators $\hat{R}^{nm}$ and $\hat{J}^{nm}$ defined as

$$\begin{aligned} \hat{R}^{nm} &\equiv (\hat{A}_+^{nm} - \hat{A}_-^{nm})/\sqrt{2} = (|n\rangle\langle m| + |m\rangle\langle n|)/\sqrt{2}, \\ \hat{J}^{nm} &\equiv (\hat{B}_+^{nm} - \hat{B}_-^{nm})/\sqrt{2} = i(|n\rangle\langle m| + |m\rangle\langle n|)/\sqrt{2}, \end{aligned} \tag{9}$$

fulfilling the orthogonality relations

$$\begin{aligned} \mathrm{Tr}\{\hat{R}^{nm}\hat{R}^{pq}\} &= (\delta_{n,p}\delta_{m,q} + \delta_{n,q}\delta_{m,p}), \\ \mathrm{Tr}\{\hat{J}^{nm}\hat{J}^{pq}\} &= (\delta_{n,p}\delta_{m,q} - \delta_{n,q}\delta_{m,p}), \\ \mathrm{Tr}\{\hat{R}^{nm}\hat{J}^{pq}\} &= 0 \end{aligned} \tag{10}$$

for $n, m, p, q = 1, \ldots, N$, where $\delta_{n,m}$ is the Kronecker delta. The set $\{\hat{R}^{mn}, \hat{J}^{mn} : n \leq m\}$ constitutes a complete basis set of N^2 operators. This operator basis gives a unique expansion of any operator $\hat{Q}$ as

$$\hat{Q} = \sqrt{2}\left(\sum_{m=2}^{N}\sum_{n=1}^{m-1} r_{nm}\hat{R}^{mn} + j_{nm}\hat{J}^{mn}\right) + \frac{1}{\sqrt{2}}\sum_{m=1}^{N} r_{mm}\hat{R}^{mm}, \tag{11}$$

with $r_{nm} = \mathrm{Tr}\{\hat{Q}\hat{R}^{mn}\}/\sqrt{2} = (Q_{mn} + Q_{nm})/2$ and $j_{nm} = \mathrm{Tr}\{\hat{Q}\hat{J}^{mn}\}/\sqrt{2} = (Q_{mn} - Q_{nm})i/2$. If $\hat{Q}$ is a Hermitian operator r_{nm} and j_{nm} are the real and imaginary parts of the matrix elements, $Q_{nm} \equiv \langle n|\hat{Q}|m\rangle$.

Fano introduced the idea of expanding the density matrix in terms of an orthogonal operator basis [1], hence we call this an "operator basis" representation. We introduced this representation for its mathematical properties rather than its physical contents. Let us note that the sensitivity optimized states mentioned before Eq. (6) can analogously be cast into this kind of orthogonal operator basis, in this sense the operator basis representation is contained in the minimal one.

Quantum optical realization.—Next we describe a possible experimental scheme for the reconstruction of a density operator describing the state of a single optical field mode [18]. It is a straightforward matter to generalize this to several optical modes. We use the *Fock state basis* in which the numbers of photons in the mode under consideration label the states $\{|m\rangle : m = 0, 1, 2, \ldots\}$. Our task is to show that the expectation values of the corresponding projection operators $\hat{A}_{\pm}^{nm}, \hat{A}$, etc. can be obtained experimentally. We note from the outset that the experimentally difficult part of the scheme at present is the preparation of coherent superpositions of two Fock states. However, in light of recent theoretical [19,20] and experimental results [21], it is clear that the problem of the preparation of the probe field can and will be solved.

Thus, since this is not a *fundamental* difficulty, we assume in the following that such superposition states are available.

The expectation value of the projection operators in the representations can be determined using the experimental setup depicted in Fig. 1 as follows.

A probe field is prepared in a particular state $|\psi\rangle$ and fed into port 1 of the beam splitter, the signal field prepared in the (unknown) state $\hat{\varrho}$ is fed into port 2. The joint photon number probability distribution of the output ports of the beam splitter is obtained from the photoelectron statistics produced in the photodetectors I and II for many repetitions of the experiment; let us note that multiphoton coincidence counts together with quantum efficiencies above 70% have been demonstrated experimentally [22,23]. If one chooses a method that detects single photons with more than 50% quantum efficiency, the photon number probability distribution can

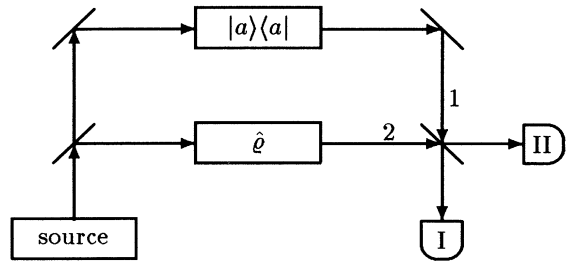

FIG. 1. The setup of our proposed quantum optical scheme. Light from a common field source is fed into a device generating the probe field $|a\rangle\langle a|$ and a device that generates the signal field $\hat{\varrho}$. The probe and signal field, which are labeled 1 and 2, respectively, are then entangled at the last beam splitter and analyzed by the photodetectors I and II. The use of a common source ensures that the probe and signal fields oscillate at the same frequency.

be recovered from the measurements using the inverse Bernoulli transformation discussed by Lee [24].

Furthermore, a new method developed by Munroe *et al.* [25] allows us to measure the photon-number statistics from the phase-averaged quadrature-field distribution with single photon and ultrahigh time resolution of the order of 300 fs. Employing the corresponding reconstruction schemes [26], this method yields almost perfect photon number statistics.

Hence we may restrict our considerations to the "true" joint photon probability distribution $P_{|\psi\rangle}(p, q)$ for p and q photons measured by (ideal) photodetectors I and II, respectively, which is given by

$$P_{|\psi\rangle}(p,q) = \sum_{n'=0}^{p+q}\sum_{m'=0}^{p+q}\langle n'|\hat{\varrho}|m'\rangle \times \langle p+q-n'|\psi\rangle\langle\psi|p+q-m'\rangle \times A_p(n', p+q-n')A_p^*(m', p+q-m'). \tag{12}$$

Here $A_p(\nu, \mu)$ represents the probability amplitude of finding mode I in the Fock state $|p\rangle_I$ if modes 1 and 2 are in the product Fock state $|\nu\rangle_1|\mu\rangle_2$ and is given by

$$A_p(\nu,\mu) = (-1)^{\nu}\sqrt{\frac{p!(\nu+\mu-p)!}{\nu!\mu!}}e^{i\varphi_\tau(p-\mu)}e^{i\varphi_\rho(p-\nu)} \times \sum_{k=0}^{\nu}\sum_{l=0}^{\mu}(-1)^k\binom{\nu}{k}\binom{\mu}{l}\sqrt{\tau^{\mu+k-l}\rho^{\nu-k+l}}\,\delta_{k+l,p}, \tag{13}$$

where τ and ρ are the transmittance and reflectance and φ_τ and φ_ρ are the corresponding phase factors generated by the beam splitter as defined by Campos, Saleh, and Teich in Ref. [27]. Inserting for $|\psi\rangle$ the special probe field states $|a^{nm}\rangle$ with $n > m$, see Eq. (1), and relabeling $p + q = N + n = M + m$ changes Eq. (12) into

$$P_{|a^{nm}\rangle}(p, N+n-p) = C\langle a'|\hat{\varrho}|a'\rangle, \tag{14}$$

where

$$|a'\rangle = C^{-1/2}[A_p^*(N,n)N_a|N\rangle + A_p^*(N,n)N_a a^*|M\rangle],$$
$$C = |A_p(N,n)N_a|^2 + |A_p(N,n)N_a a|^2 . \quad (15)$$

Again we assume that the diagonal elements are known, for example, by the comparatively simple measurement of the photocount distribution of the field alone. The same is assumed to be true for a, which is known from the state preparation process, we can thus, equivalently to Eq. (3), use $P_{|a^{nm}\rangle}(p, N + n - p)$ to determine a quantity

$$M_{|a^{nm}\rangle}(N,p) \equiv 2\,\mathrm{Re}\{a\varrho_{MN}A_p(M,m)A_p^*(N,n)\}, \quad (16)$$

where "Re" signifies the real part. Using a second linearly independent probe state $|b^{nm}\rangle$, by a procedure analogous to Eqs. (4) and (5) we obtain ϱ_{MN}. Thus *we have translated the minimal representation into an experimental scheme in quantum optics for determining the quantum state of light;* the translation of the other representations along similar lines is straightforward.

It is interesting to note that the value of p in Eq. (14) can be chosen arbitrarily from the interval $(0 \le p \le N + n)$. This gives $N + n + 1$ different ways of determining the value of the quantity $M_{|a^{nm}\rangle}(N,p)$ in Eq. (16). Also, since we require $n - m = M - N$ in Eq. (14) the set of matrix elements $\varrho_{(k+N-M)k}$ for $k = 0, 1, 2, \ldots$ can be determined from just two probability distributions $P_{|a^{nm}\rangle}$ and $P_{|b^{nm}\rangle}$ for fixed values of n and m. And finally, since it is only the difference $n - m$ that decides which set of matrix elements are determined, this implementation is also redundant in the sense that the probe states $|a^{st}\rangle$ with $s = t + n - m$ are equivalent for $t = 0, 1, 2 \ldots$.

This scheme will give as many matrix elements of the density operator as desired and is limited only by experimental error and the ability to prepare the probe field in suitable two-Fock-state superpositions.

We examined the requirements for representing any density operator in terms of expectation values of simple projection operators. We gave two different representations: the minimal representation, which requires the least number of projectors, and the operator basis representation, which gives the expansion of any operator in terms of an operator basis. Our results are applicable to any physical system whose state space is of countable dimension N which need not be finite.

We showed how the expectation values could be determined experimentally for the case of a single mode of an optical field. An important point about our method is that it is *not recursive,* in contrast to some other methods for determining the density operator of the optical field [6,12,13] for which the calculation of all but a select few matrix elements involves the previously calculated values of other matrix elements and results in the accumulation of experimental errors.

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[15] The first constraint results from the requirement that $m_{|a\rangle}$ in Eq. (4) shall be least sensitive to variations of a, i.e., $\partial(aN_a^2)^{-1}/\partial a = 0$, and the same for b. The second stems from the requirement that the overall multiplication factor of the vector $(m_{|a\rangle}, m_{|b\rangle})$ i.e., the determinant of $\boldsymbol{T}$ in Eq. (5), shall be least sensitive to variations of the relative phase $\beta - \alpha$. This implies $\beta - \alpha$ to be odd multiples of $\pi/2$ or $b = \pm ia$.
[16] A projector is defined as $\hat{P}^2 = \hat{P}$ and so, strictly speaking, $\hat{A}_+^{nn}/2$ is a projector but $\hat{A}_+^{nn}$ is not. However, for brevity, we also refer to the latter as a projector in this paper.
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